Chiral edge states in topological insulators and superconductors

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Outline

0) Preliminaries of (topological) band theory

I) Integer quantum Hall effect

II) The anomalous quantum Hall effect

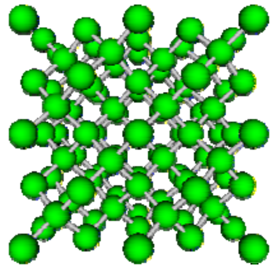
III) A brief incursion into 2D topological insulators

IV) 2D chiral topological superconductors

V) How about 3D ?

Preliminaries of (topological) band theory

Elements of Traditional Band Theory



Non-interacting electrons moving in a perfectly periodic array of atoms

- Electron Hamiltonian commutes with lattice translations

$$[\mathrm{H}, \mathrm{T}\,(\mathbf{R})] = 0$$

$$\boldsymbol{R} = n_x \boldsymbol{a}_x + n_y \boldsymbol{a}_y + n_z \boldsymbol{a}_z,\ \ n_\alpha \text{ is an integer}$$

Lattice translation Symmetry

$$\mathrm{T}(\boldsymbol{R})|\psi_{\boldsymbol{k}}\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{R}}|\psi_{\boldsymbol{k}}\rangle$$

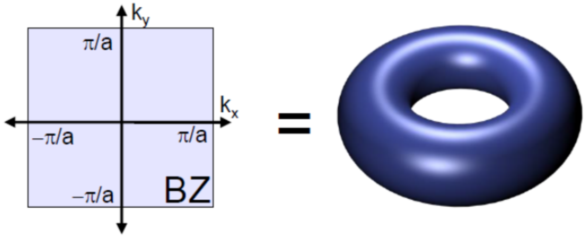
Crystal momentum ***k*** *is conserved*

- The wave vector k is defined modulo the reciprocal lattice vector (reciprocal lattice is the Fourier transform of the real-space lattice)

$$\boldsymbol{k} \sim \boldsymbol{k}\ mod\ \boldsymbol{G}$$

- The wave-vector ***k*** "lives" on a d-dimensional torus

$$\boldsymbol{k} \in \mathbb{T}^d\ \ (1D: -{}^{\pi}/_{a} \le \boldsymbol{k} \le {}^{\pi}/_{a}, \text{ with the end points "glued")}$$



Elements of Traditional Band Theory

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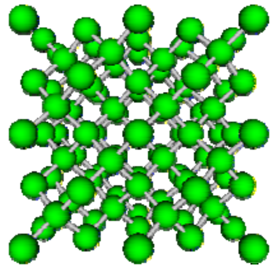

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Crystal momentum ***k*** *is conserved*

Bloch thm:

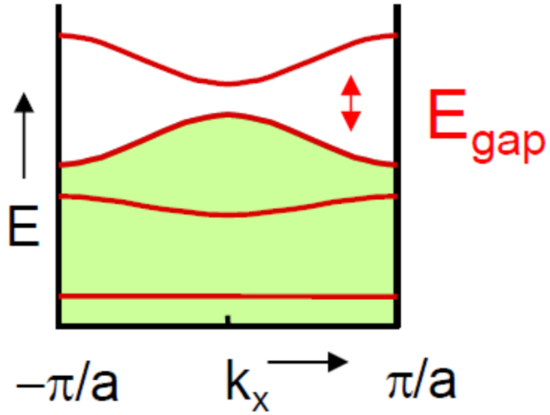
$$|\psi_{\boldsymbol{k}}\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{R}}|u_{\boldsymbol{k}}\rangle$$

of period of **a**

Bloch Hamiltonian:

$$H(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{r}}\mathrm{H}e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$H(\mathbf{k})|u_n(\mathbf{k})\rangle = E_n(\mathbf{k})|u_n(\mathbf{k})\rangle$$

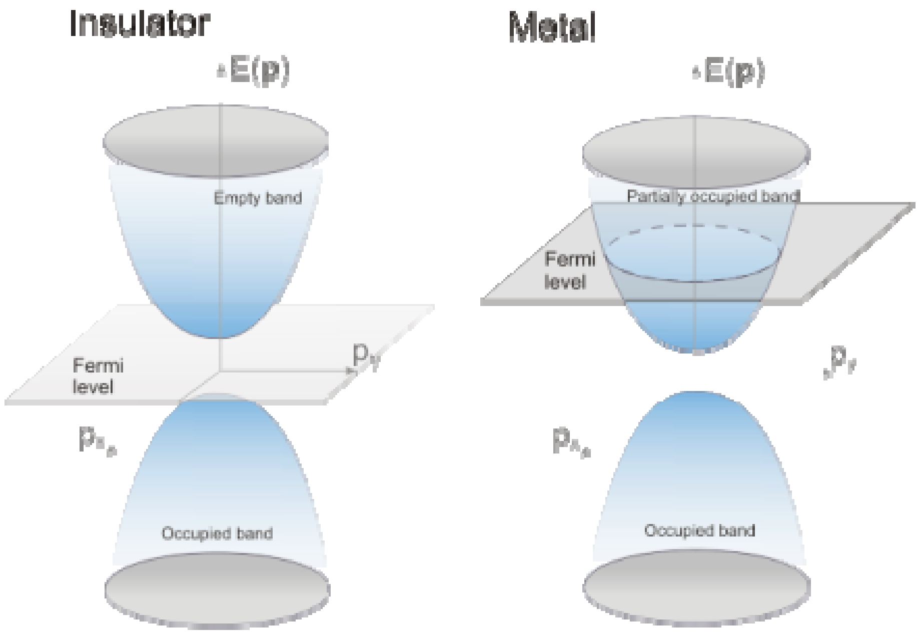


Insulators and metals

- Bloch theorem and band structure:

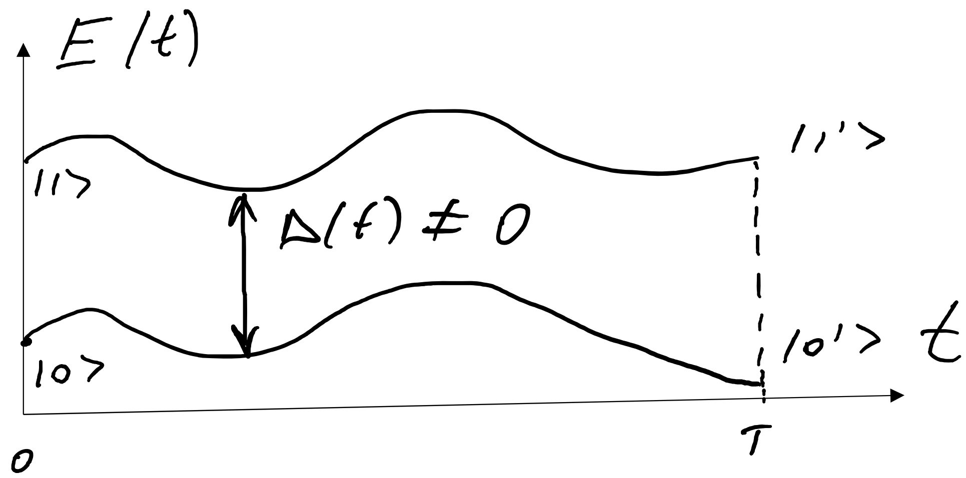
$$\left[-\frac{\hbar^2\nabla^2}{2m}+V(\mathbf{r})\right]\psi(\mathbf{r})=E\psi(\mathbf{r}), V(\mathbf{r})=V(\mathbf{r}+\mathbf{a})$$

$$\psi_{\mathbf{p}}(\mathbf{r})=u_{\mathbf{p}}(\mathbf{r})\,e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}},\ \text{with } u_{\mathbf{p}}(\mathbf{r})=u_{\mathbf{p}}(\mathbf{r}+\mathbf{a})$$



Quantum topological equivalence

- How to define topological invariants for quantum states of matter?
- We need a notion of topological equivalence of quantum states.
- The notion of quantum topological equivalence follows from adiabatic continuity



If we can adiabatically deform $|0\rangle$ into $|0'\rangle$, then $|0\rangle \sim |0'\rangle$

Band topological equivalence

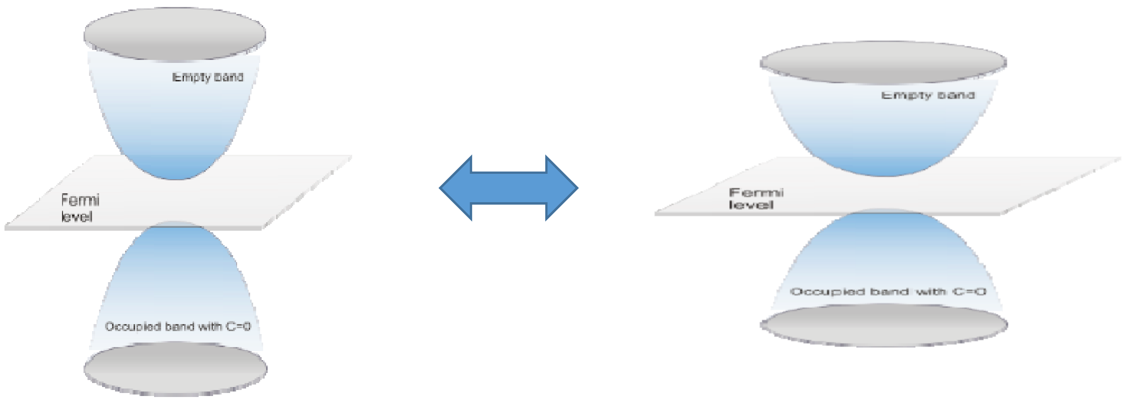
- How to define topological invariants for quantum states of matter?
- We need a notion of topological equivalence of quantum states.

Topological Equivalence : adiabatic continuity

Band structures are equivalent if they can be continuously deformed into one another without closing the energy gap

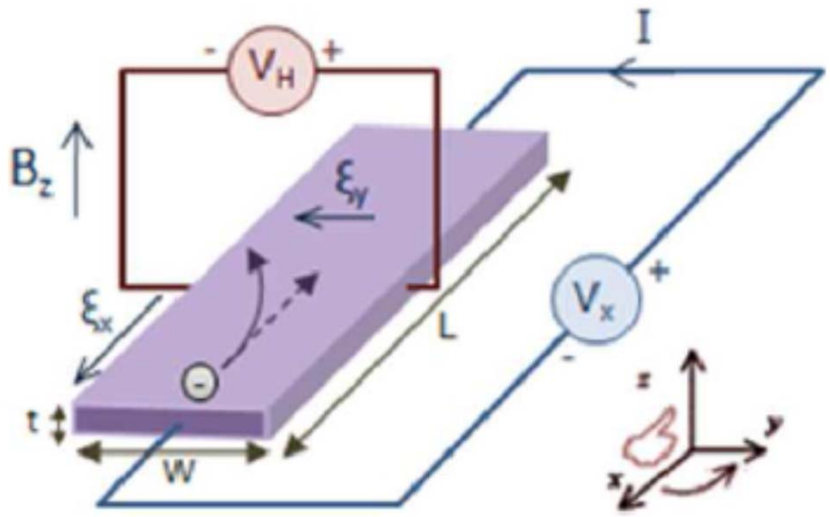
Insulator I

Insulator II



I) The integer Quantum Hall effect

Classical Hall effect (1879)



Classical equation of motion

$$m\left(\frac{d\vec{v}}{dt}+\frac{\vec{v}}{\tau}\right)=-e(\vec{E}+\vec{v}\wedge\vec{B})$$

Conductivity tensor

$$\sigma_{xx} = \frac{\sigma_0}{1+(\omega_c\tau)^2}$$
$$\sigma_{yx} = \frac{\sigma_0}{1+(\omega_c\tau)^2}$$

Drude Conductivity

$$\sigma_0 = \frac{ne^2\tau}{m}$$

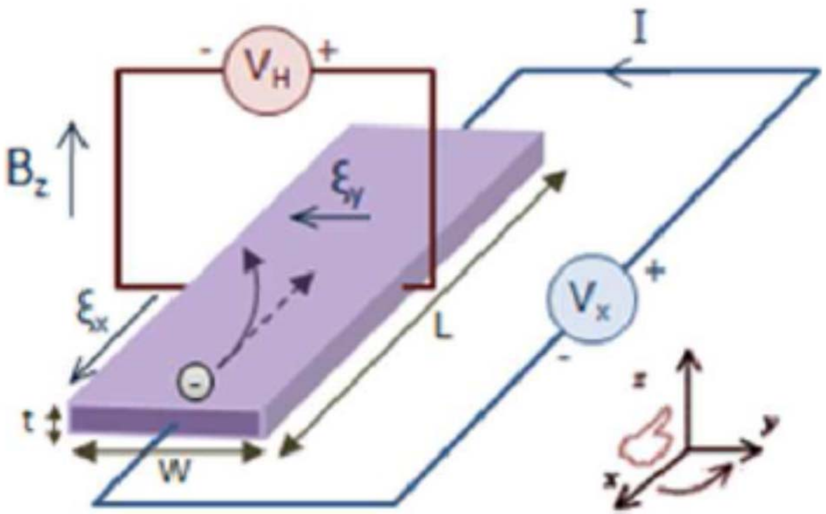
$$\omega_c = \frac{eB}{m}$$

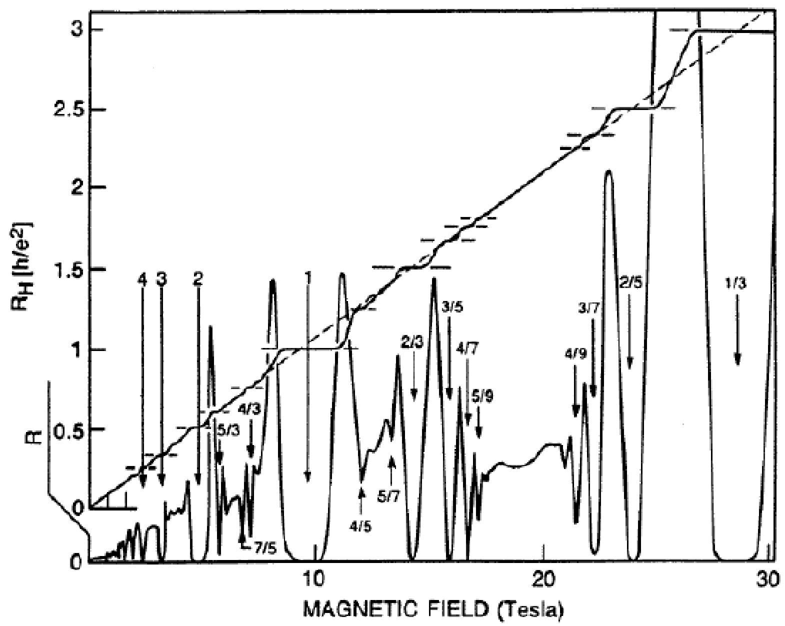
Resistivity tensor

$$\rho_{xx} = \sigma_0^{-1}$$
$$\rho_{xy} = \frac{m\omega_c}{ne^2} = \frac{1}{ne}B.$$

$$\rho_H = \rho_{xy} \propto B$$

Quantum Hall effect (1980)

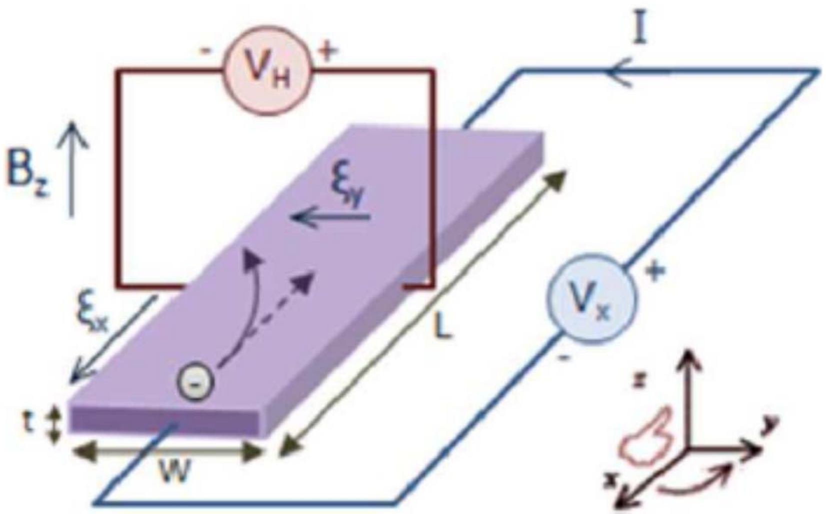


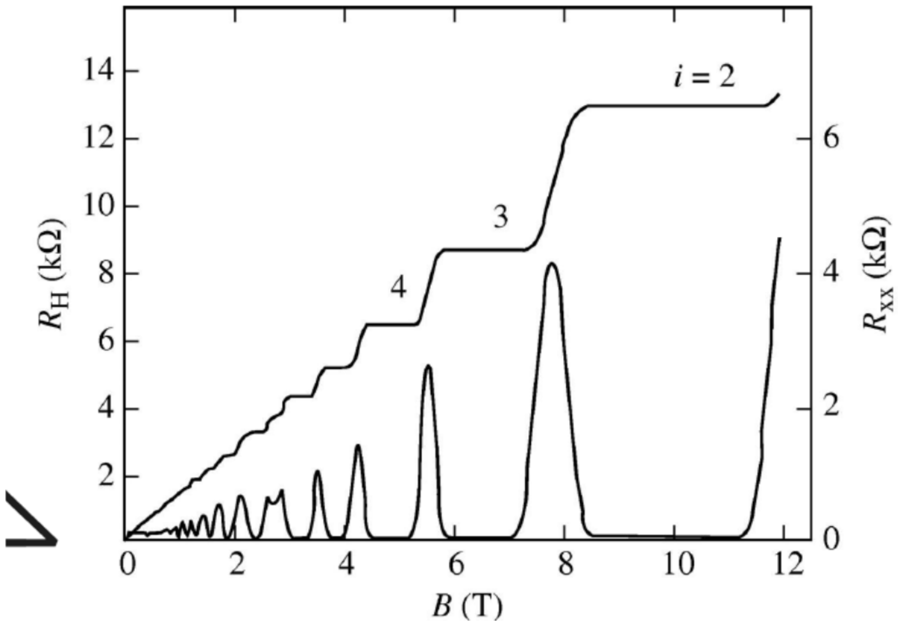


Stormer, *Physica* **B177**, 401 (1992)

K. v. Klitzing, G. Dorda, and M. Pepper, PRL 45, 494 (1980)

Quantum Hall effect





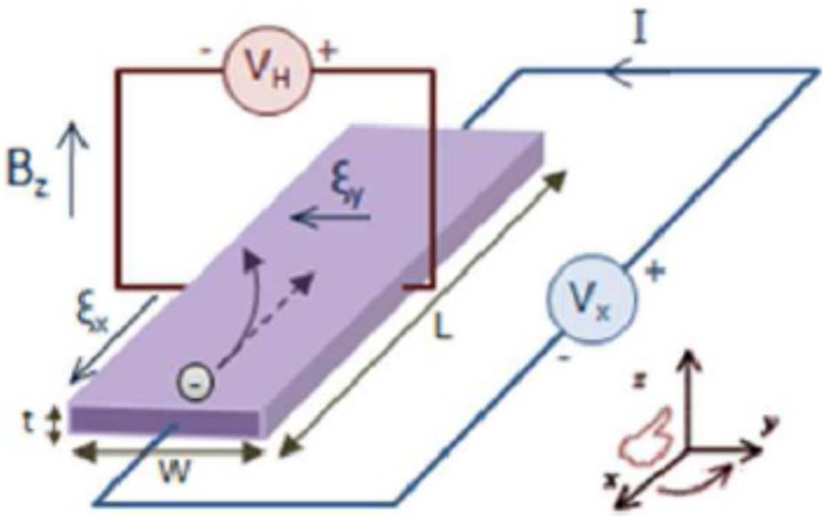
K. v. Klitzing, G. Dorda, and M. Pepper, PRL 45, 494 (1980)

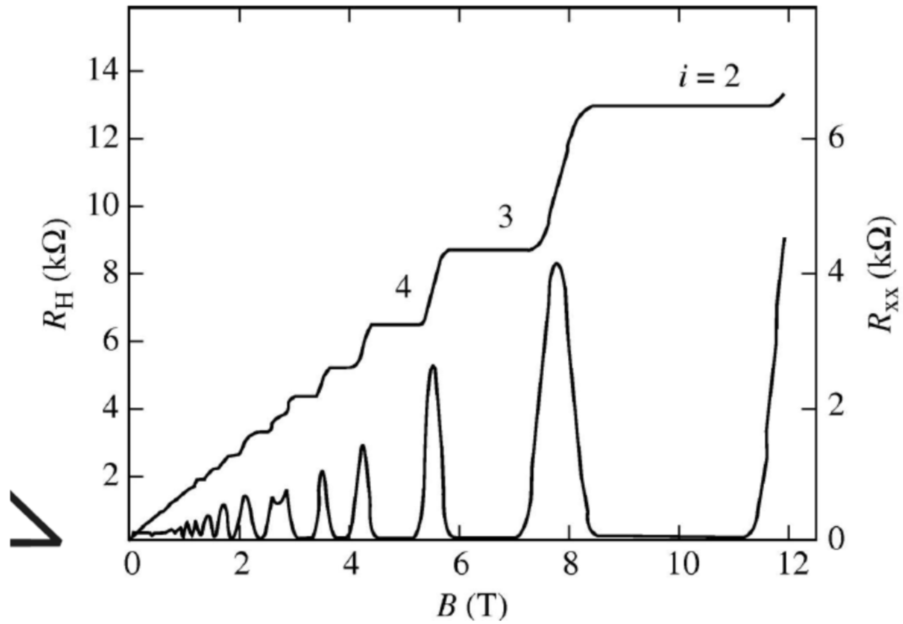
- **Quantization** of the Hall resistance at low temperature : $R_H = \frac{h}{e^2}\frac{1}{n}$ Results independent of geometrical and microscopic details

$R_K = \frac{e^2}{h} \approx 25812.807\,\Omega$ Quantum of resistance; UNIVERSAL constant

Used as a metrological unit : help to redefine the unit of mass !

Quantum Hall effect





K. v. Klitzing, G. Dorda, and M. Pepper, PRL 45, 494 (1980)

→ Quantum Hall conductivity changes by plateaus.

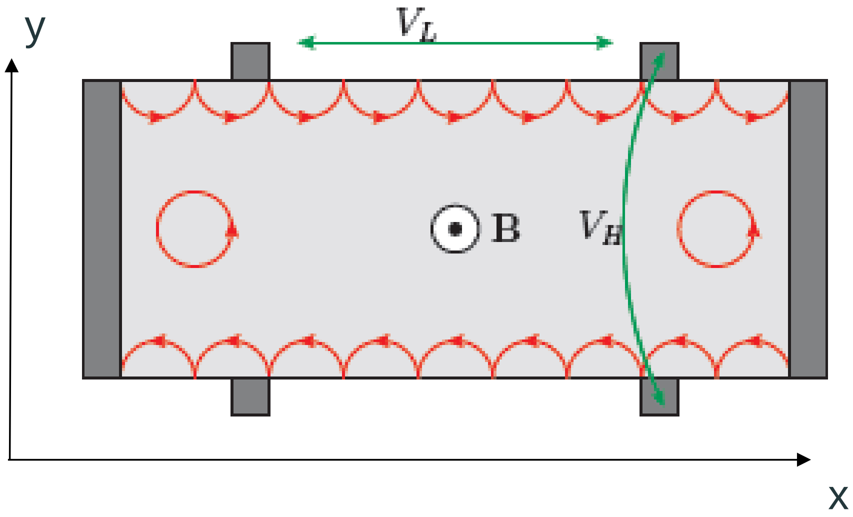
→ Each plateau is perfectly quantized by an integer number in unit of e^2/h

$$J_y = \sigma_{xy} E_x$$

$$\sigma_{xy} = n \frac{e^2}{h}$$

Integer accurate to 10^{-9}

Semi-classical picture

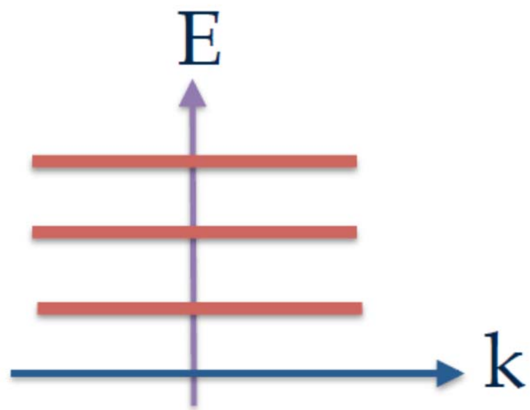


2D Cyclotron Motion,
Landau Levels

Electron in an orbital magnetic field :

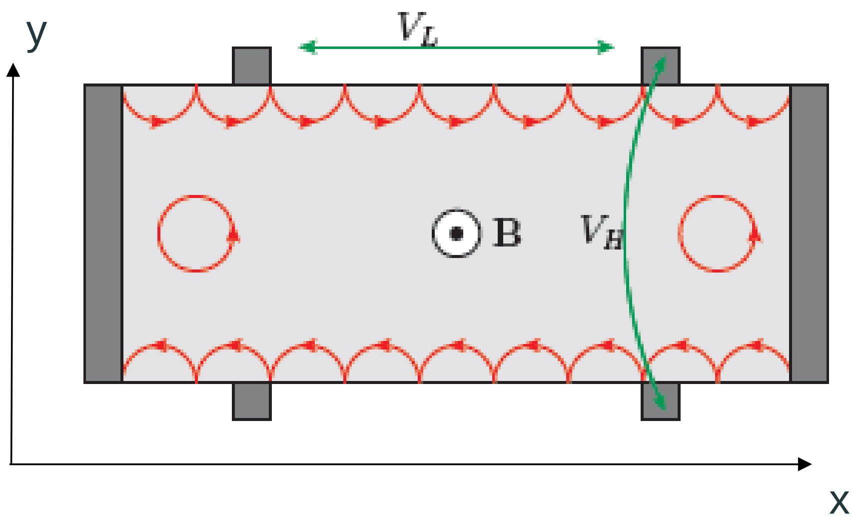
$$H = \frac{1}{2m_e}(\vec{p} + e\vec{A})^2$$

$$\varepsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega_c,$$ Landau levels

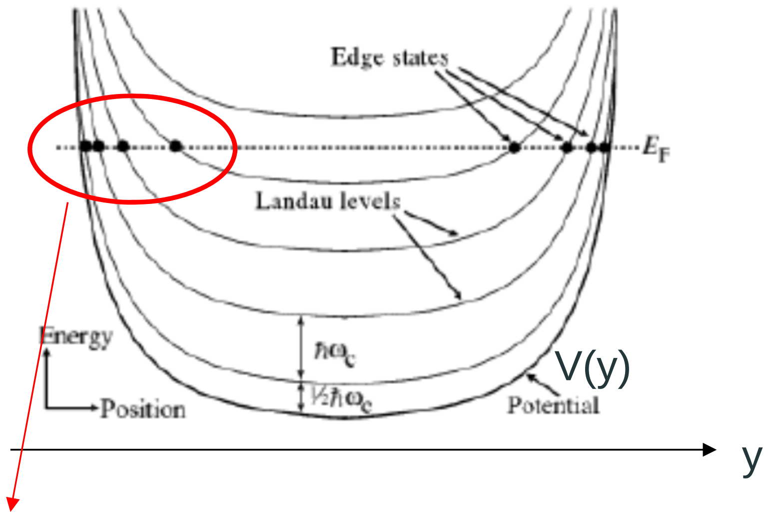


Why such perfect robustness & quantization ?

Semi-classical picture



2D Cyclotron Motion,
Landau Levels

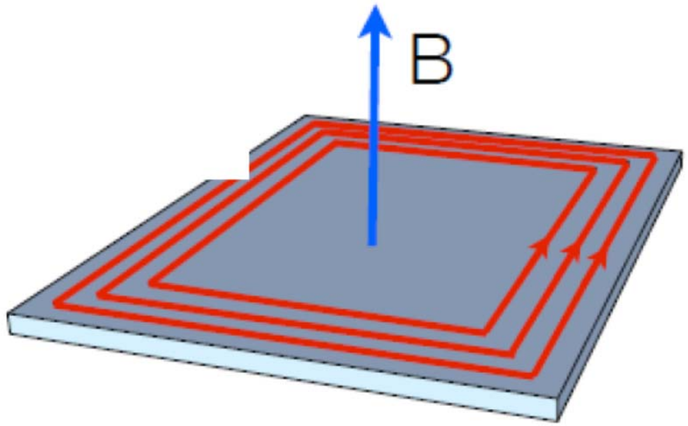


Edge states= skipping orbits

- Landau levels (LLs) bend near sample edge.
- The Fermi level intersects LLs at the edge.
- Nb of edge states at the Fermi level= Nb of occupied bulk LLs

Landau levels with a bulk gap and (protected) edge states

The edges' viewpoint: Robustness of n



- Electrons on same edge move along the same direction.

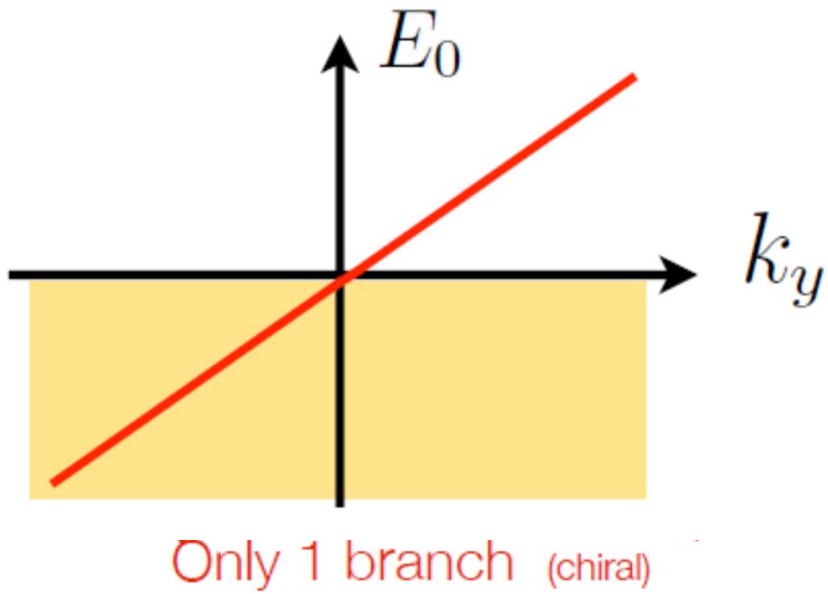
- Electrons on opposite edges move along the opposite directions.

Chirality = Consequence of time reversal symmetry breaking

Robustness against backscattering

- chiral edge state cannot be localized by disorder (no backscattering)

- edge states are therefore **perfect charge conductors**



The bulk point of view

The quantum Hall effect: a topological property?

Distinction between the integer quantum Hall state and a conventional insulator is a topological property of the band structure

$\mathcal{H}(\mathbf{k})$: Brillouin zone $\longmapsto$ Hamiltonians with energy gap

Classified by Chern number: $n = \frac{i}{2\pi} \sum_{\substack{\text{filled} \\ \text{states}}} \int \mathcal{F} d^2k$ (= topological invariant) $n \in \mathbb{Z}$

Kubo formula : $$\sigma_{xy} = \frac{e^2}{h} \frac{i}{2\pi} \sum_{\substack{\text{filled} \\ \text{states}}} \int \mathcal{F} d^2k = \frac{e^2}{h} n$$

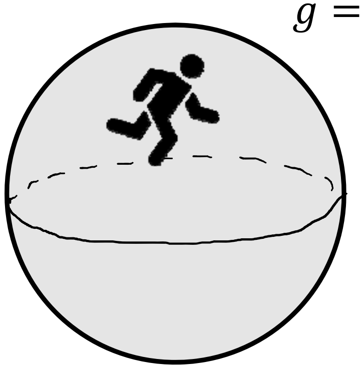
Thouless et al., PRL 49, 405 (1982)

Alternative description: n is a bulk topological invariant

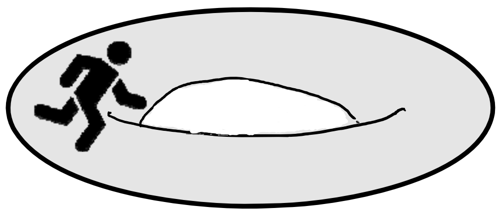
Example of a topological invariant

Can we tell by local measurements whether we are living on the surface of a sphere or a torus?

$g = 0$



$g = 1$



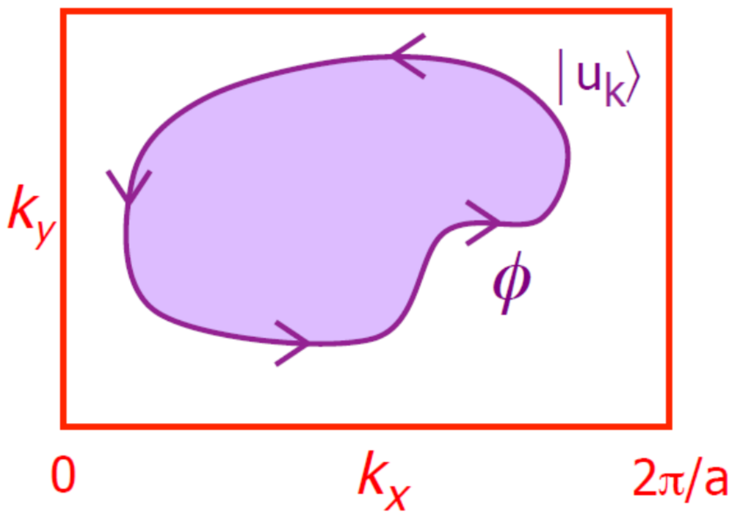
$$\int_S K_g \, dA = 4\pi(1-g)$$

Gaussian curvature, $\boldsymbol{K_g} = \pm \frac{\mathbf{1}}{\boldsymbol{R_1 R_2}}$

<u>**Topological invariant**</u> = quantity that does not change under continuous deformation

Berry connection & curvature

For a given band, we can introduce :



$$u_{\mathbf{k}}(\mathbf{r}) = e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{\mathbf{k}}(\mathbf{r})$$

Berry connection:

$$\mathbf{A}(\mathbf{k}) = -\mathrm{Im}\,\langle u_{\mathbf{k}}|\nabla_{\mathbf{k}}|u_{\mathbf{k}}\rangle$$

Berry phase :

$$\phi = \oint \mathbf{A}(\mathbf{k}) \cdot d\mathbf{k}$$

Berry curvature

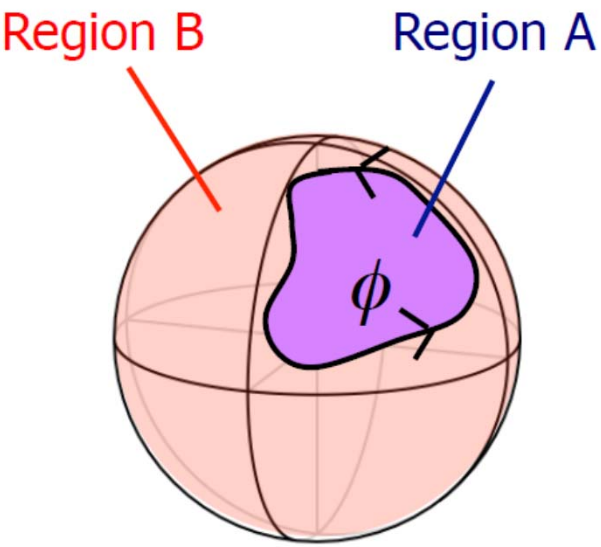
$$\Omega(\mathbf{k}) = \nabla \times \mathbf{A}$$

$$\Omega_z(\mathbf{k}) = -2\mathrm{Im}\left\langle \frac{du}{dk_x} \middle| \frac{du}{dk_y} \right\rangle$$

Stokes thm :

$$\phi = \int \Omega_z(\mathbf{k})\, d^2k$$

Chern theorem



Berry curvature

$(F \equiv \Omega)$

Stokes thm applied to A:

$$\phi = \int_A \mathcal{F}(\lambda)\, dS_\lambda \mod 2\pi$$

Stokes applied to B:

$$\phi = -\int_B \mathcal{F}(\lambda)\, dS_\lambda \mod 2\pi$$

Subtract:

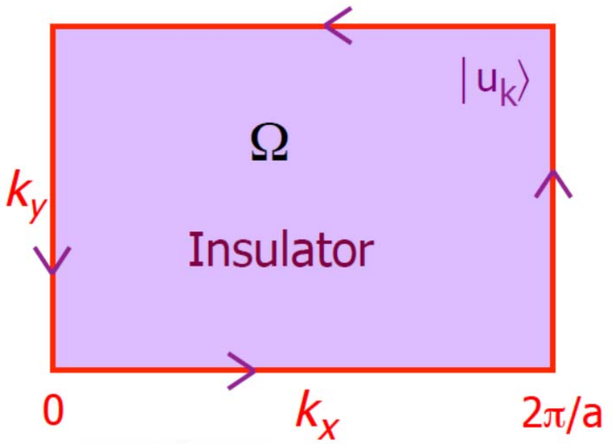
$$0 = \oint \mathcal{F}(\lambda)\, dS_\lambda \mod 2\pi$$

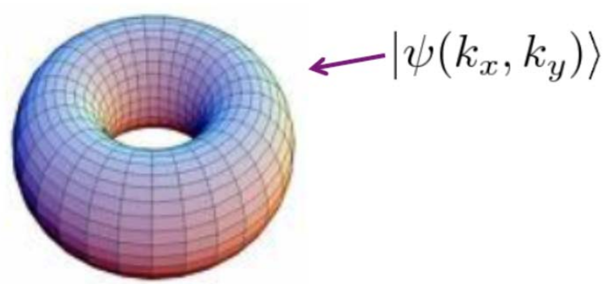
Chern Theorem: $$\oint \mathcal{F}(\lambda)\, dS_\lambda = 2\pi\, C$$ with C $\in \mathbb{Z}$

C = First Chern number

Application of Chern theorem

Let us apply this result to the Brillouin zone





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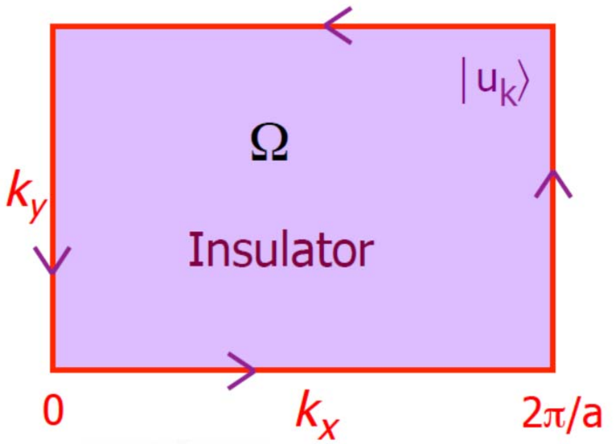
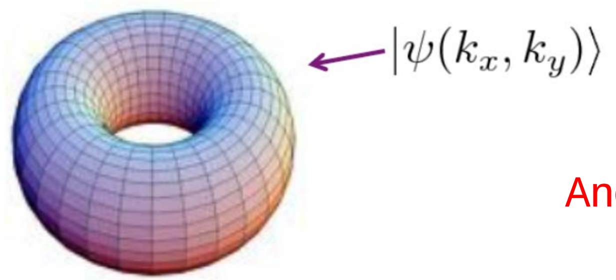


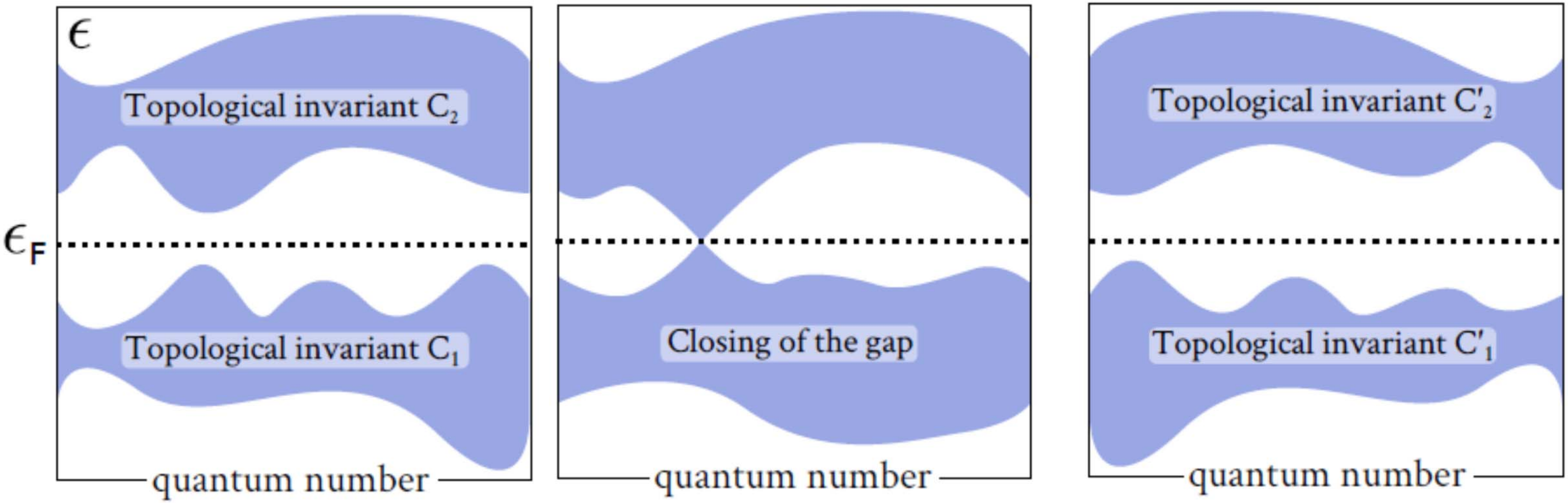


$$\Omega_z(\mathbf{k}) = -2\mathrm{Im}\left\langle \frac{du}{dk_x} \middle| \frac{du}{dk_y} \right\rangle$$

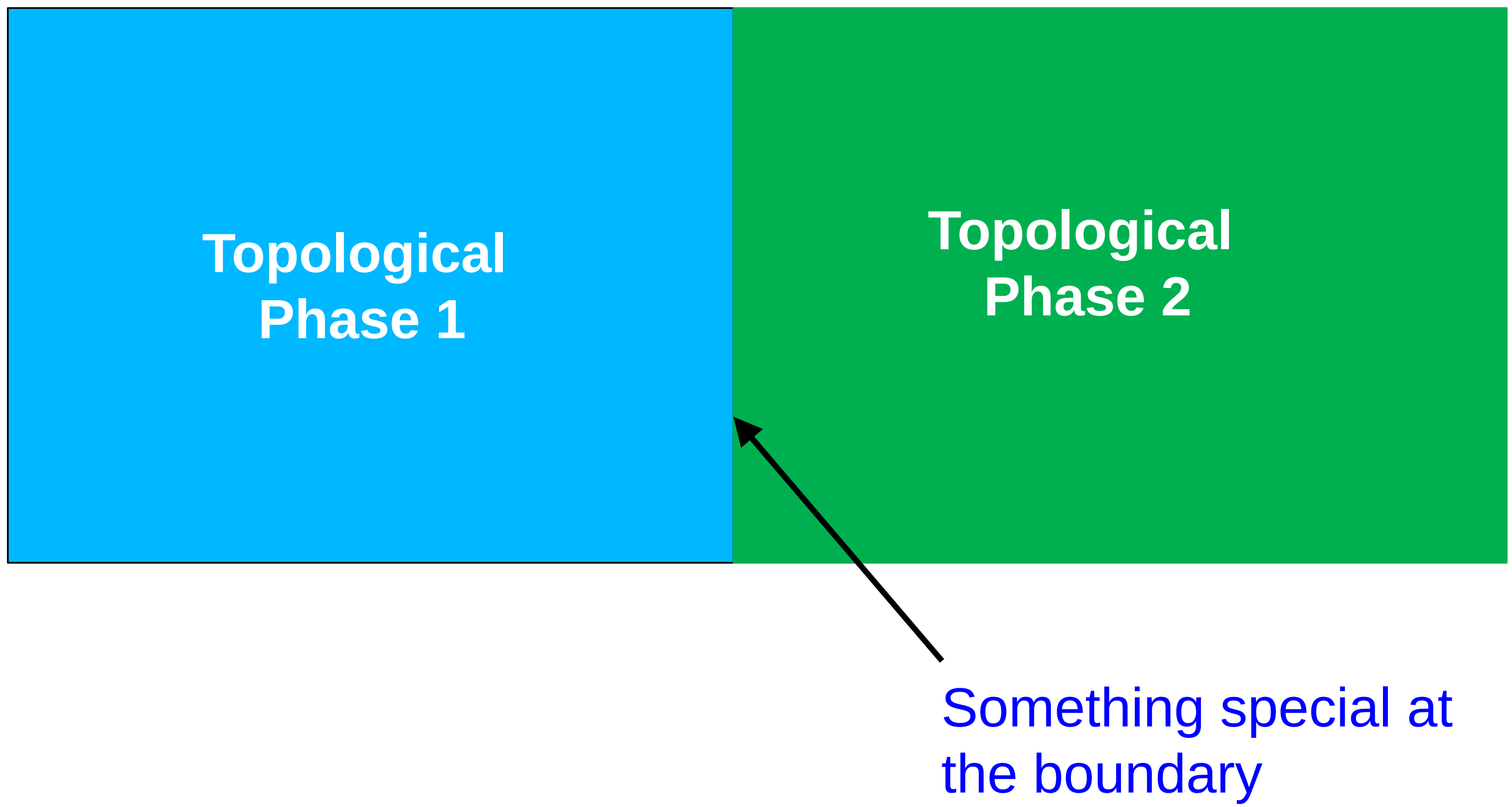
$$\phi = \int_{BZ} \Omega_z(\mathbf{k})\, d^2k = 2\pi C$$

Anomalous Hall conductivity: $\sigma_{xy} = \frac{e^2}{h} C$

Topological phase transition



Bulk-edge correspondence



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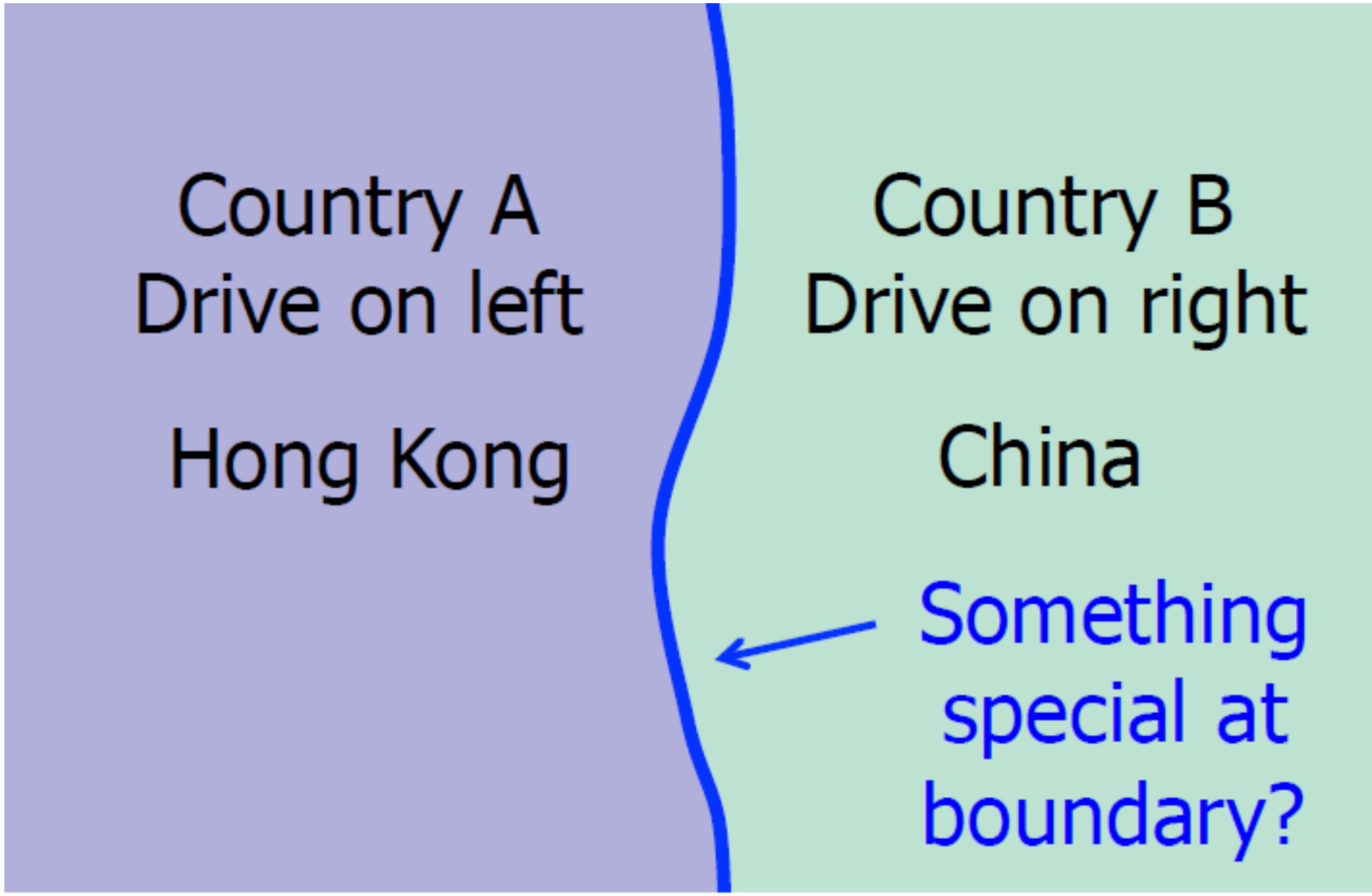
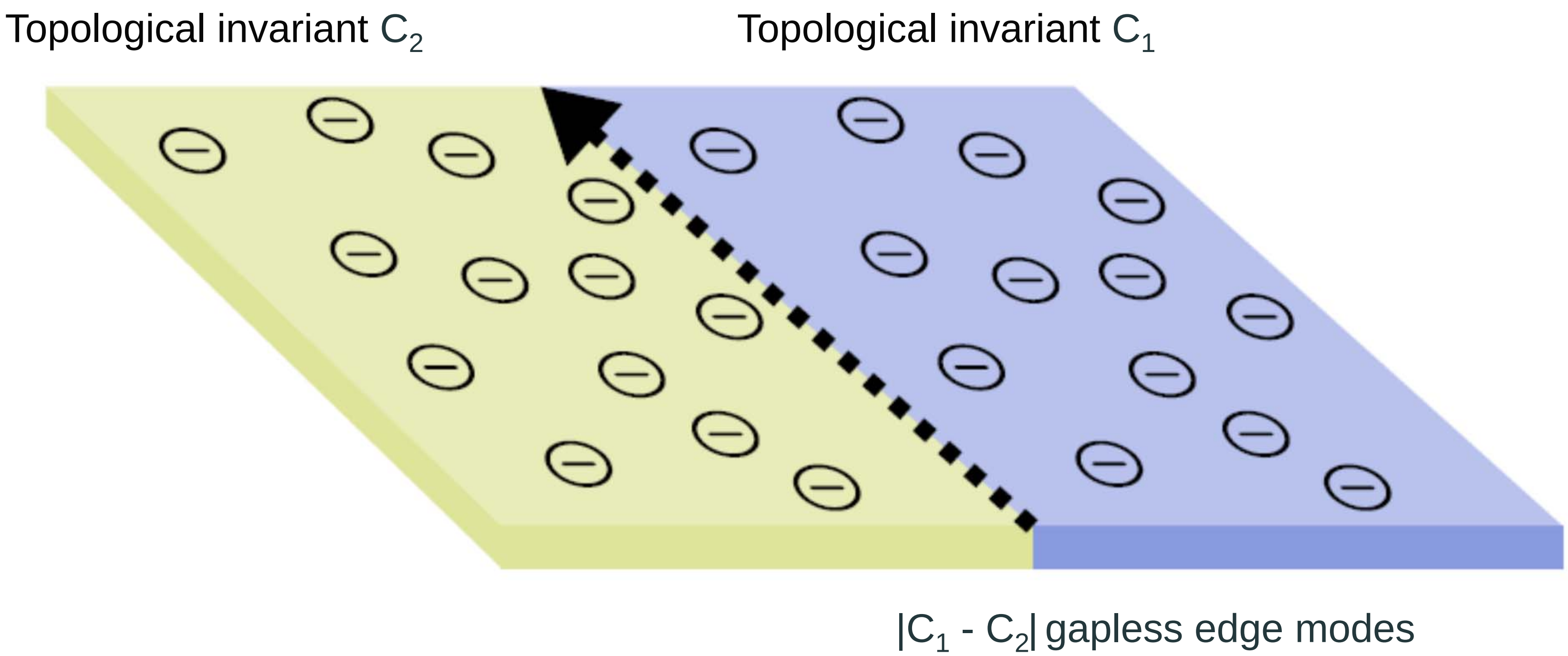


Bulk-edge correspondence

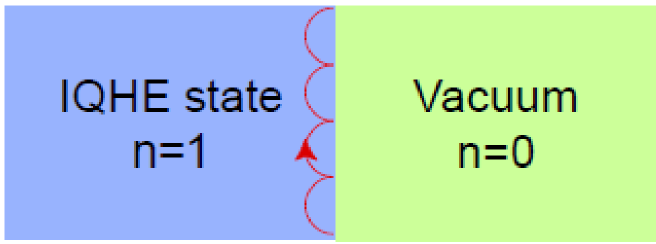
Bulk-edge correspondence

Two materials described by different topological invariants C_1 and C_2 placed in contact ➔ emergence of $|C_1 - C_2|$ gapless edge modes

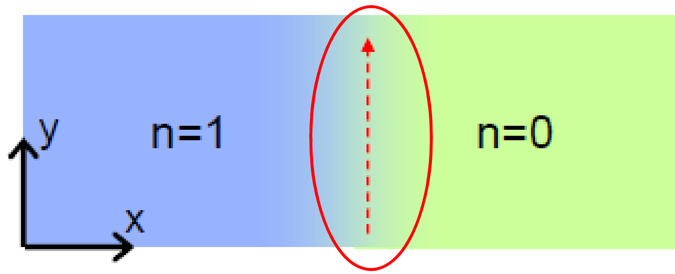


Chiral edges states in the QHE

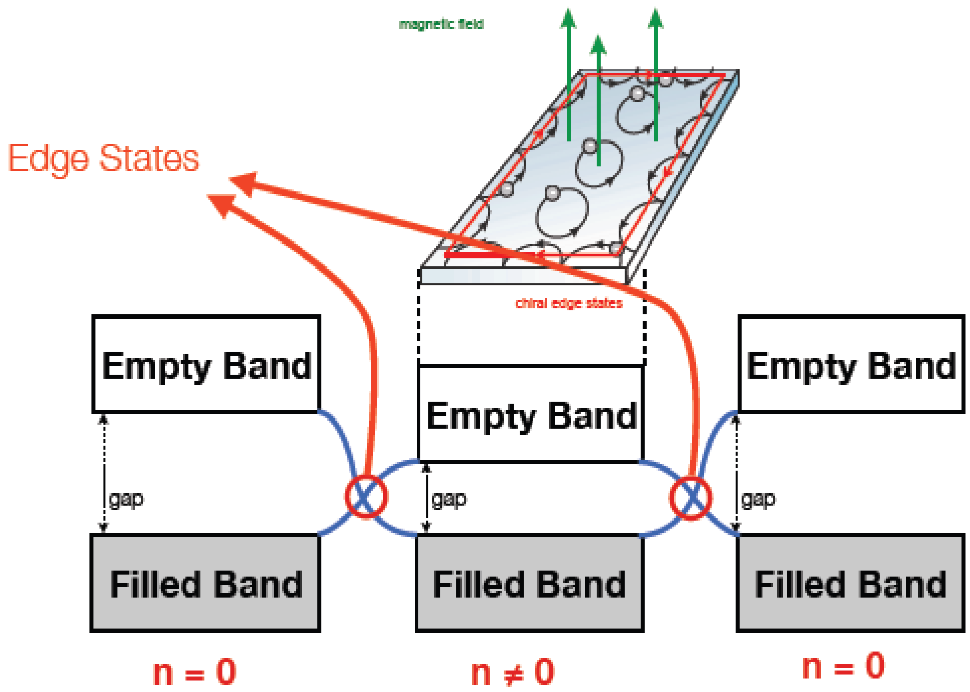
Gapless states must exist at the interface between different topological phases



Edge states ~ skipping orbits

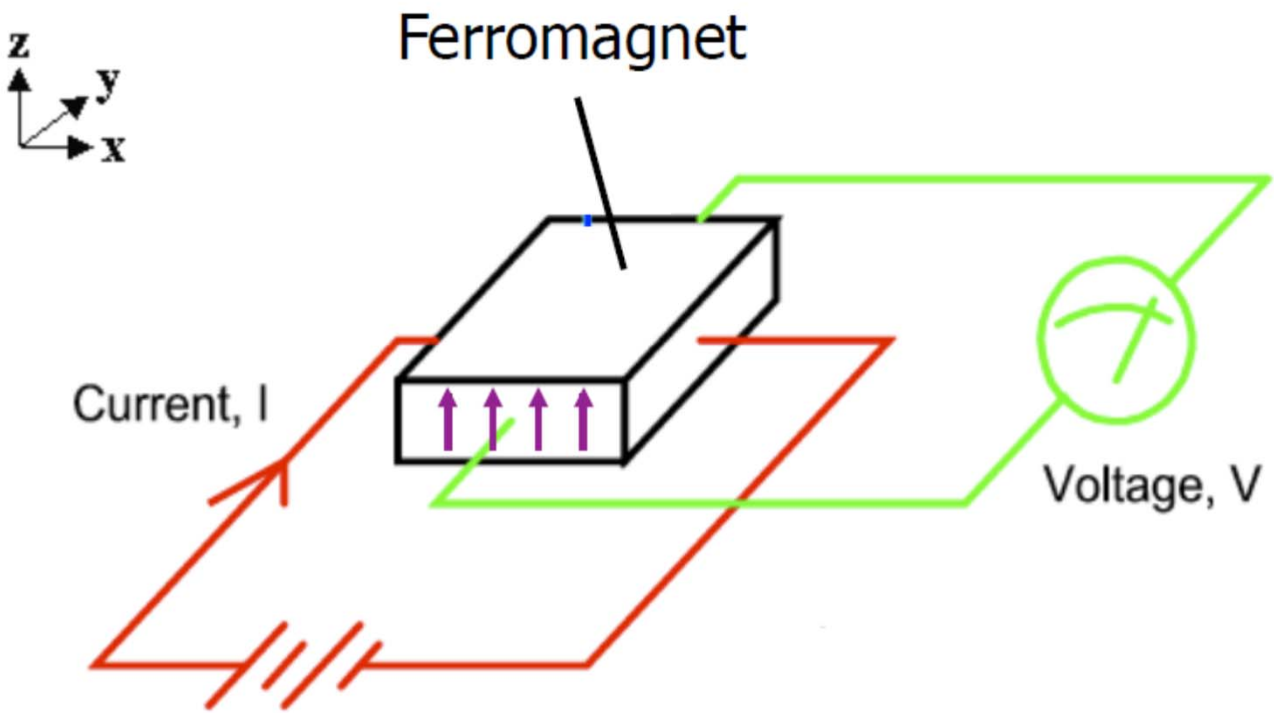


Smooth transition : band inversion



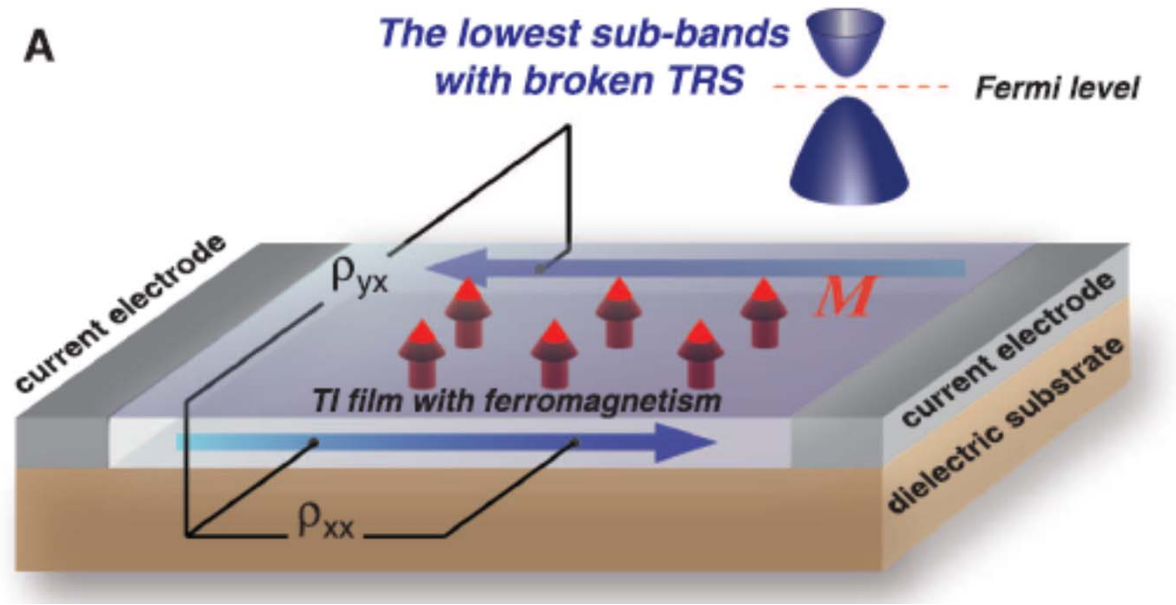
II) The anomalous quantum Hall effect or the 2D Chern insulator

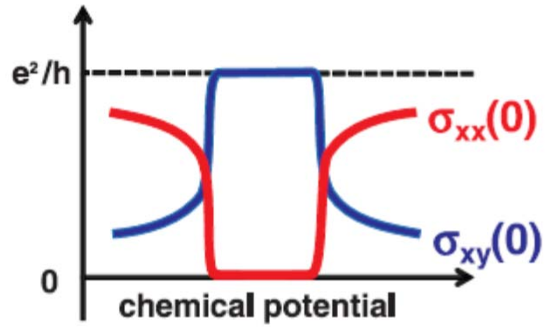
Anomalous Hall effect (1881)

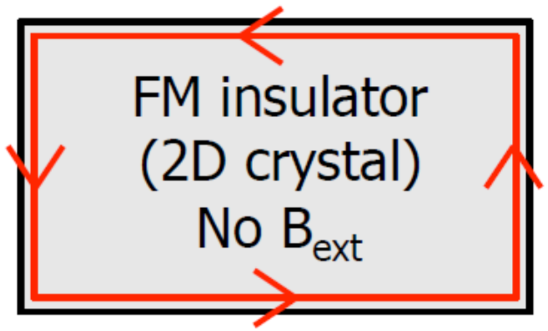


Measure of Hall conductivity in absence of a magnetic field

Quantum anomalous Hall effect (2013 ?)



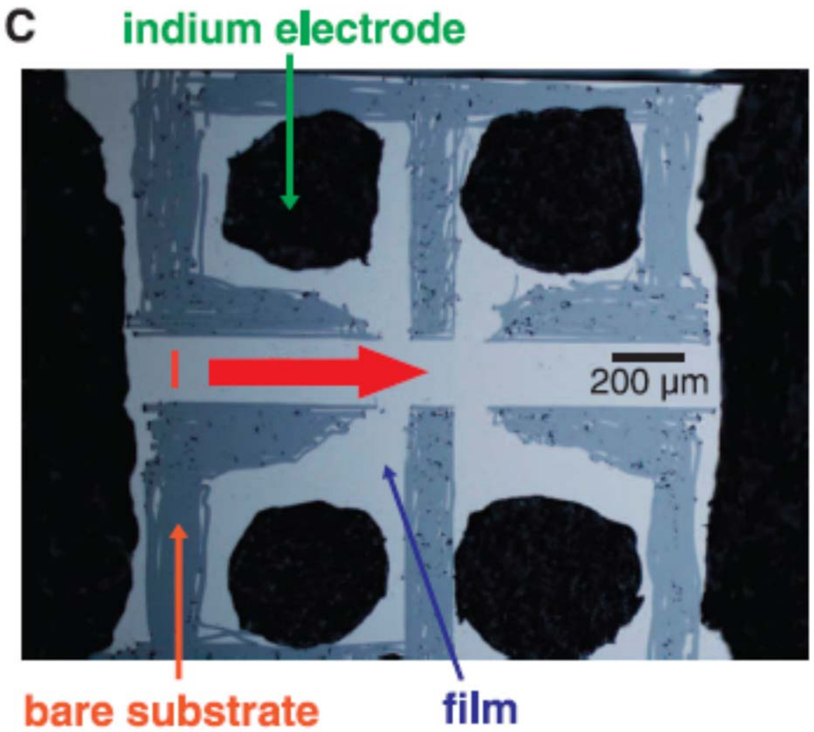


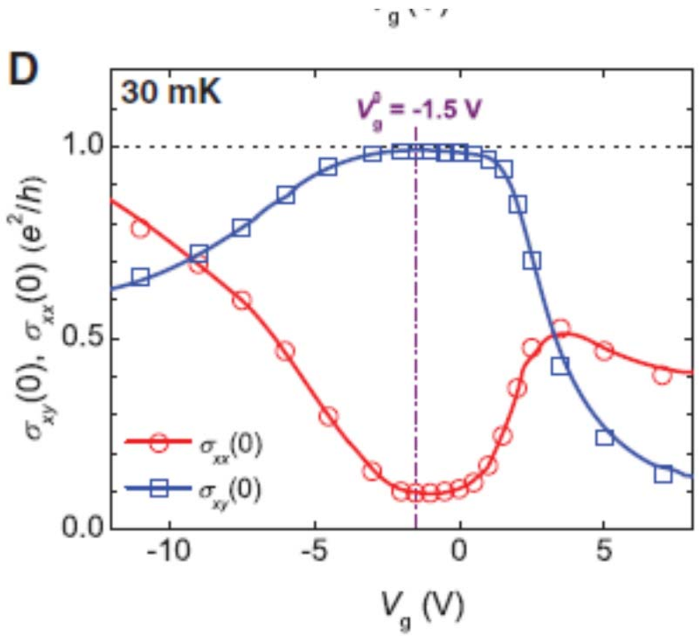


Anomalous Hall conductivity $\sigma_{xy} = \frac{e^2}{h} C$

Like integer quantum Hall effect, but no Bext

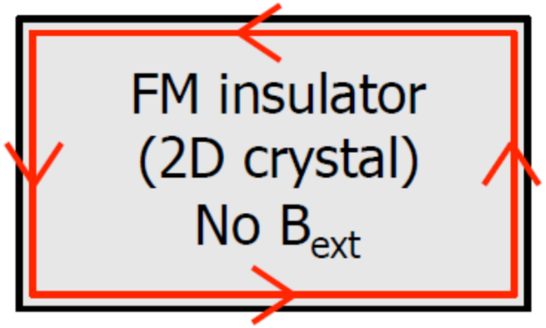
Quantum anomalous Hall effect (2013 ?)

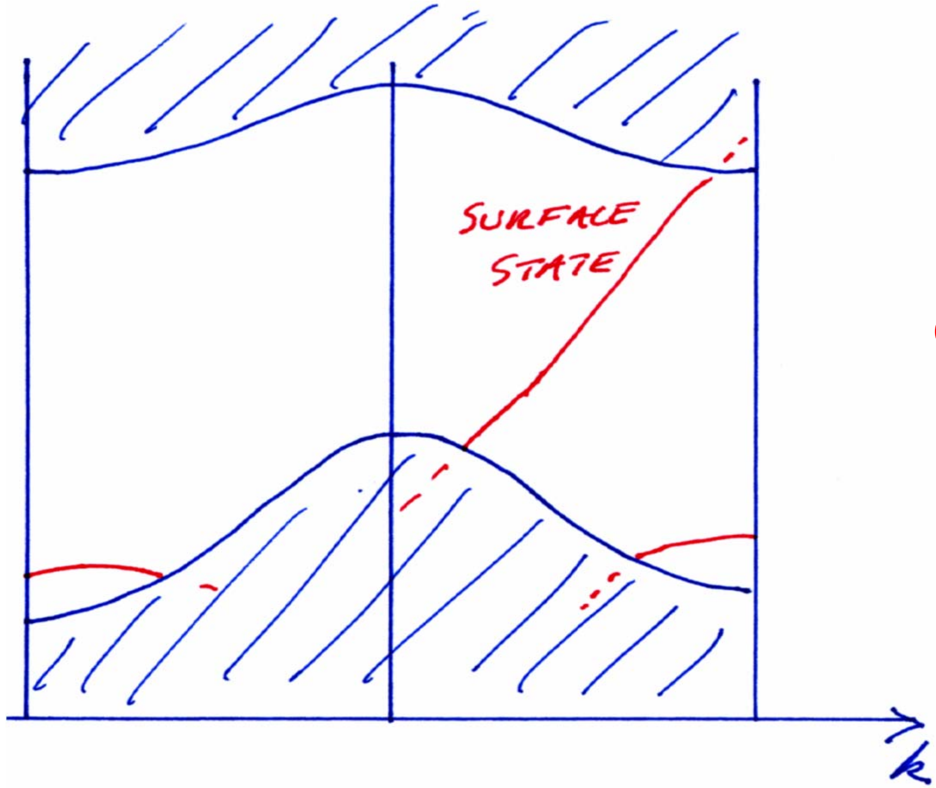




C.-Z. Zhang et al., Science 340, 167 (2013)

Edge states: 2D QAH insulator

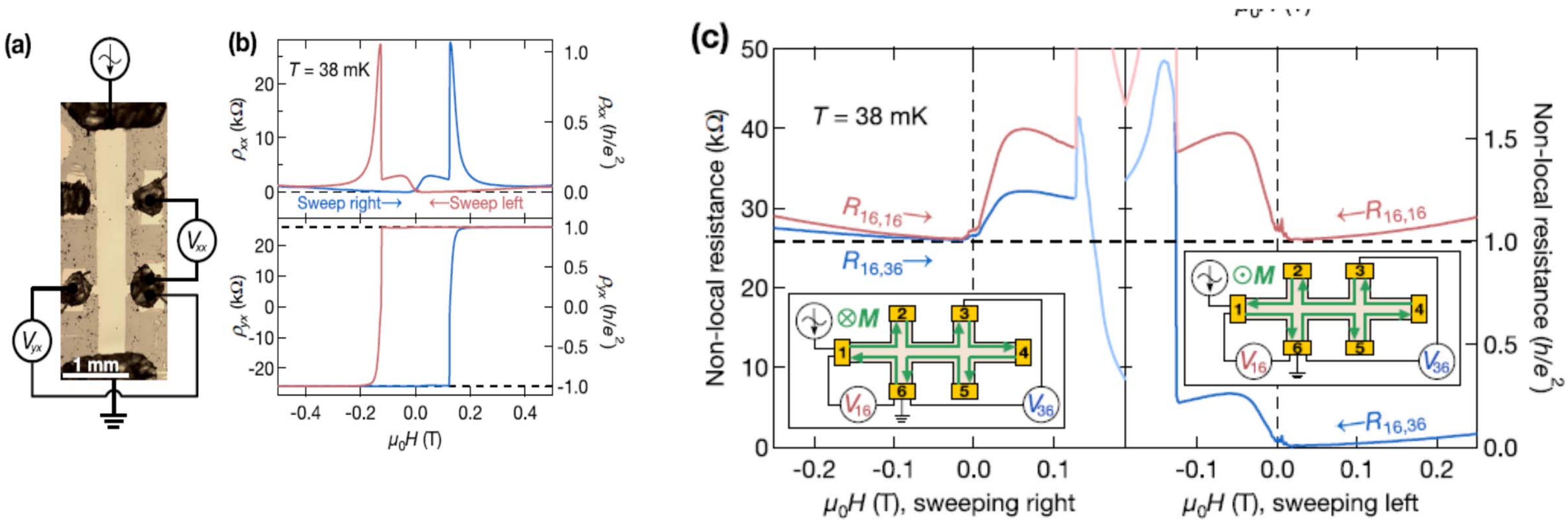




C = +1

Existence of a chiral edge state without magnetic field !

Edge states: 2D QAH insulator



A. J. Bestwick et al., PRL 114, 187201 (2015)

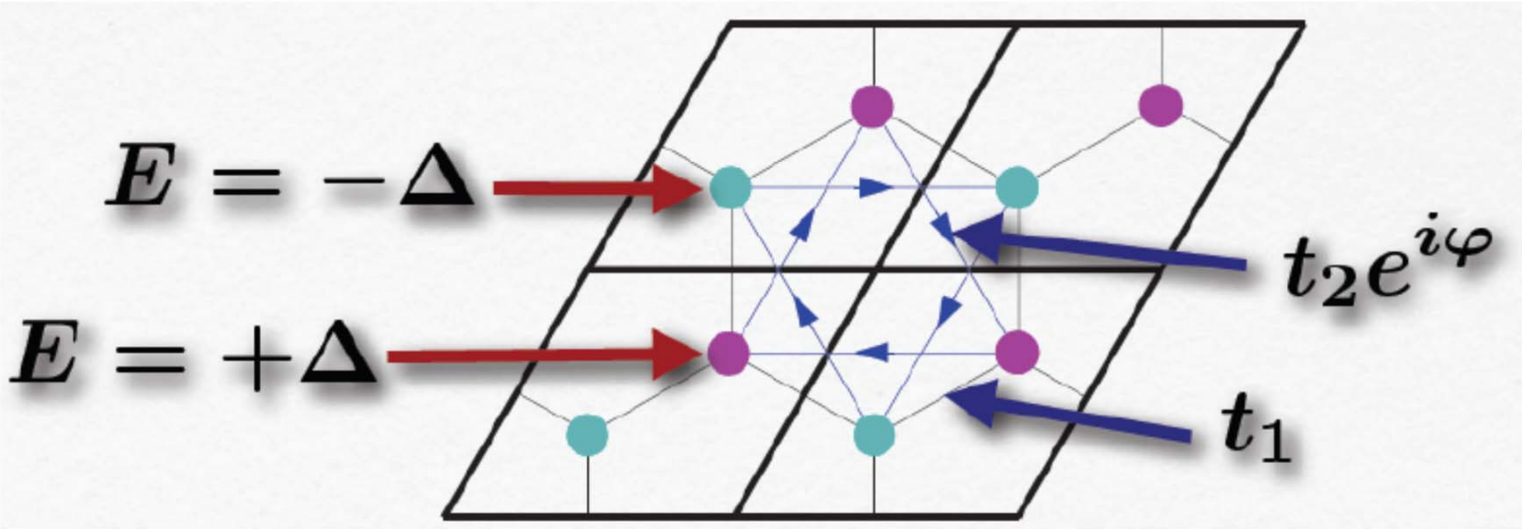
Proof of principle: the Haldane model

VOLUME 61, NUMBER 18 PHYSICAL REVIEW LETTERS 31 OCTOBER 1988

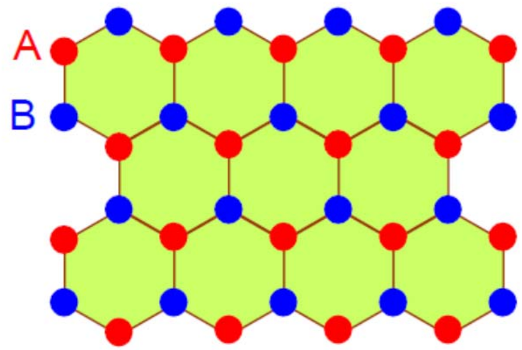
Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

F. D. M. Haldane
Department of Physics, University of California, San Diego, La Jolla, California 92093
(Received 16 September 1987)

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.



Graphene



One orbital per site

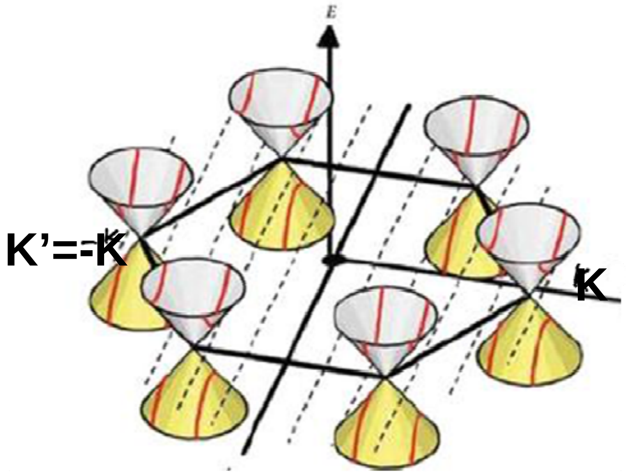
Two atoms per unit cell (A and B)

Spinless

Band structure near Dirac cones

A/B sublattice

$$h(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \sigma$$



Emergence of massless Dirac fermions at low energies:

K / K' valley

$$h(\mathbf{q}) \approx v\tau^z \sigma^x q_x + v\sigma^y q_y$$

Momentum measured from Dirac node

Symmetries of graphene

- Inversion symmetry A sublattice ⟺ B sublattice

$$\hat{\mathcal{P}} = \sigma_x \tau_x$$

$$\hat{\mathcal{P}} h(\mathbf{q}) \hat{\mathcal{P}} = h(-\mathbf{q})$$

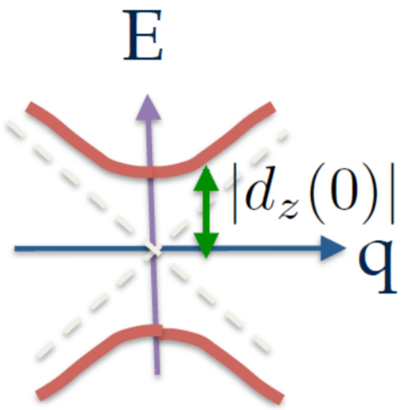
- Time reversal symmetry:

$$\hat{\mathcal{T}} = \tau_x \mathcal{K}$$

$$\hat{\mathcal{T}} h(\mathbf{q}) \hat{\mathcal{T}} = h(-\mathbf{q})$$

Making graphene insulating

$$h(\mathbf{q}) = v\,\tau^z \sigma^x\, q_x + v\,\sigma^y q_y + d_z(\mathbf{q})\sigma^z$$



Need to break either time-reversal symmetry or inversion symmetry

(i) Break inversion symmetry

$$d_z(\mathbf{q}) = m_S$$

Semenoff insulator (1984)

(ii) Break time-reversal symmetry

$$d_z(\mathbf{q}) = m_H \tau^z$$

Haldane insulator (1988)
= Quantum spin Hall insulator
= Chern insulator

Proof of principle: the Haldane model

VOLUME 61, NUMBER 18 PHYSICAL REVIEW LETTERS 31 OCTOBER 1988

Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

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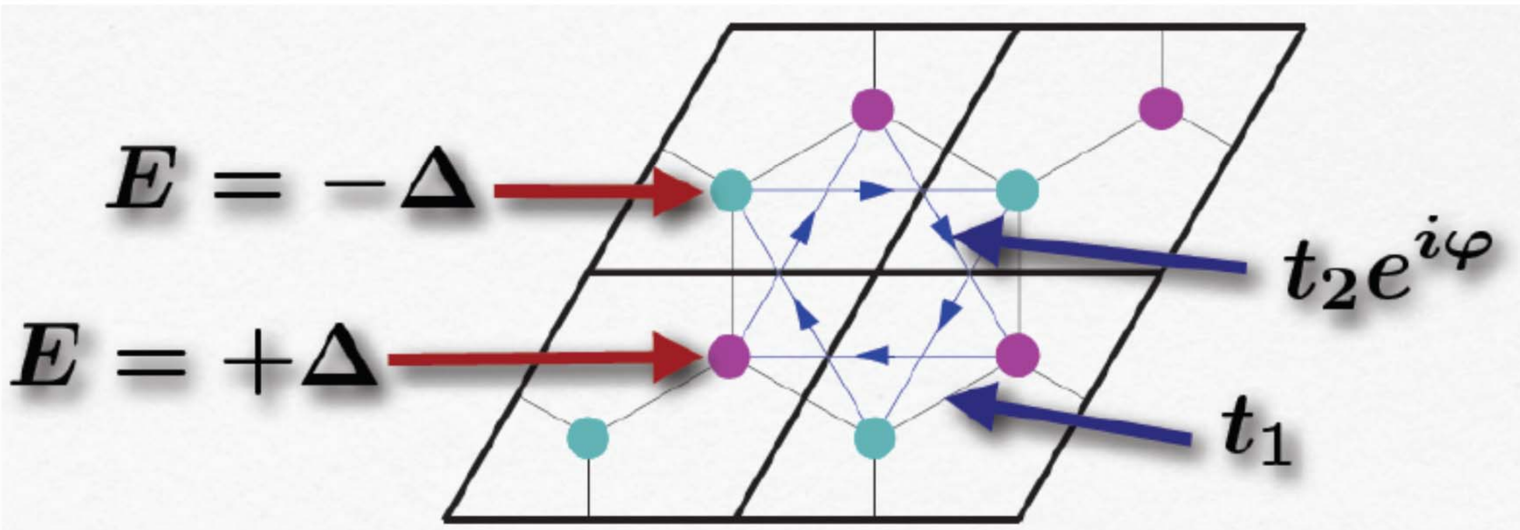
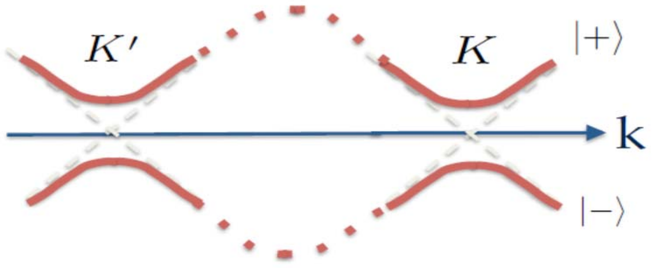
(Received 16 September 1987)

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.



Topological characterization

$$h(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \sigma$$



Two strategies:

i) Compute the eigenvectors, Berry connection, Berry phase and Chern number.

ii) Look at **d(k)**

$E_\pm = \pm|\mathbf{d}(\mathbf{k})|$ Spectrum flattening $\hat{\mathbf{d}}(\mathbf{k}) = \frac{\mathbf{d}(\mathbf{k})}{|\mathbf{d}(\mathbf{k})|}$

Mapping: $\hat{\mathbf{d}}(\mathbf{k})$: Brillouin zone $\longmapsto \hat{\mathbf{d}}(\mathbf{k}) \in S^2$

$$\text{“}\pi_2(S^2) = \mathbb{Z}\text{”}$$

Topological characterization

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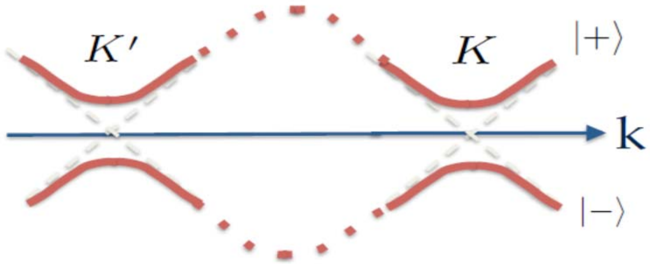

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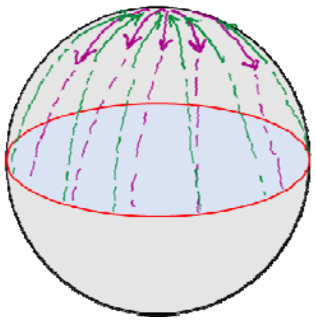
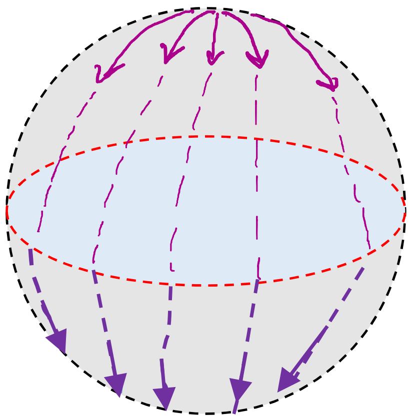
Semenoff insulator



Trivial insulator: $m_K = m_{K'}$

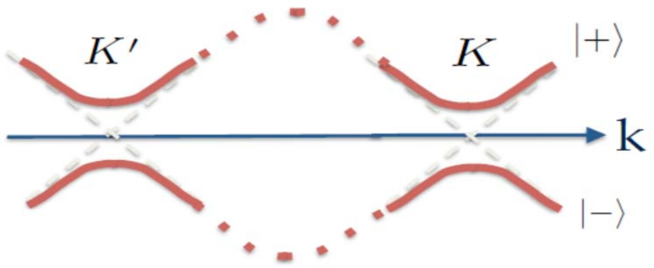
Haldane insulator



$m_K = -m_{K'}$

Topological characterization

$$h(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \sigma$$



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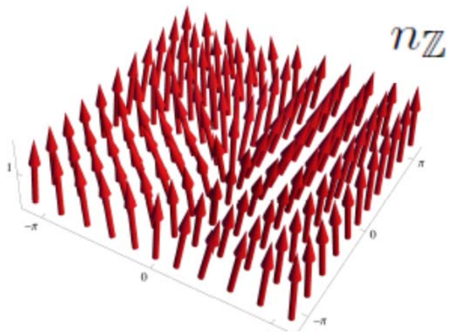
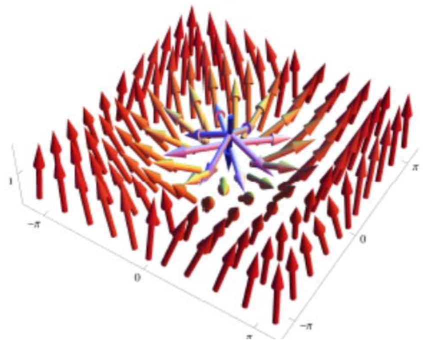
trivial phase



$n_{\mathbb{Z}} = 0$

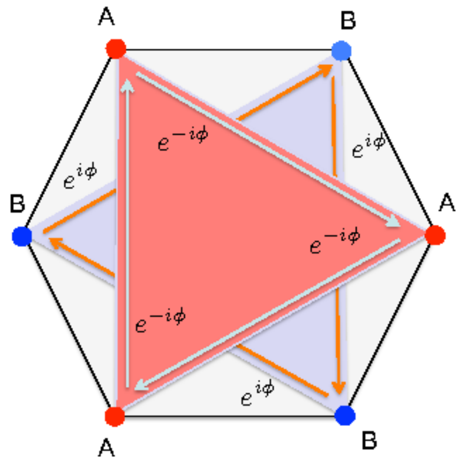
non-trivial phase

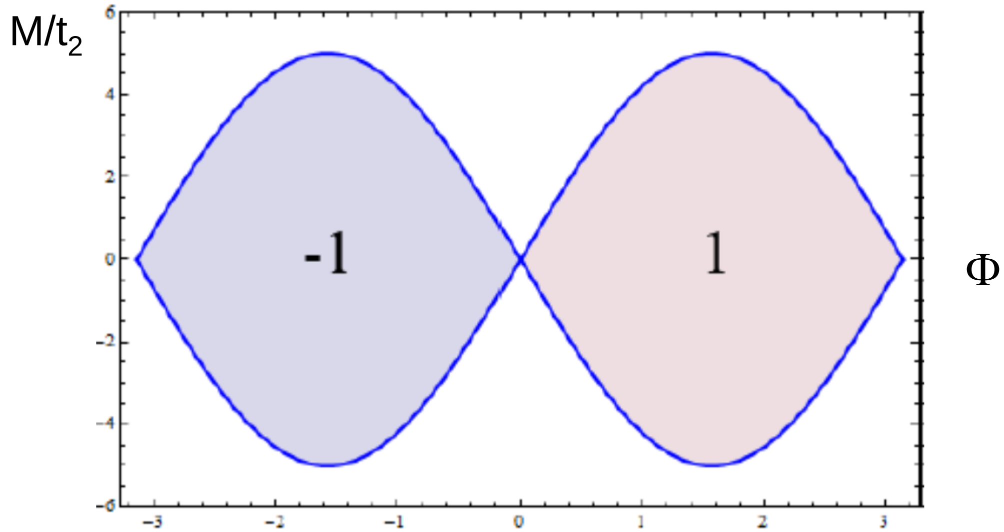


$n_{\mathbb{Z}} = \pm 1 = \mathrm{sgn}(m_H)$

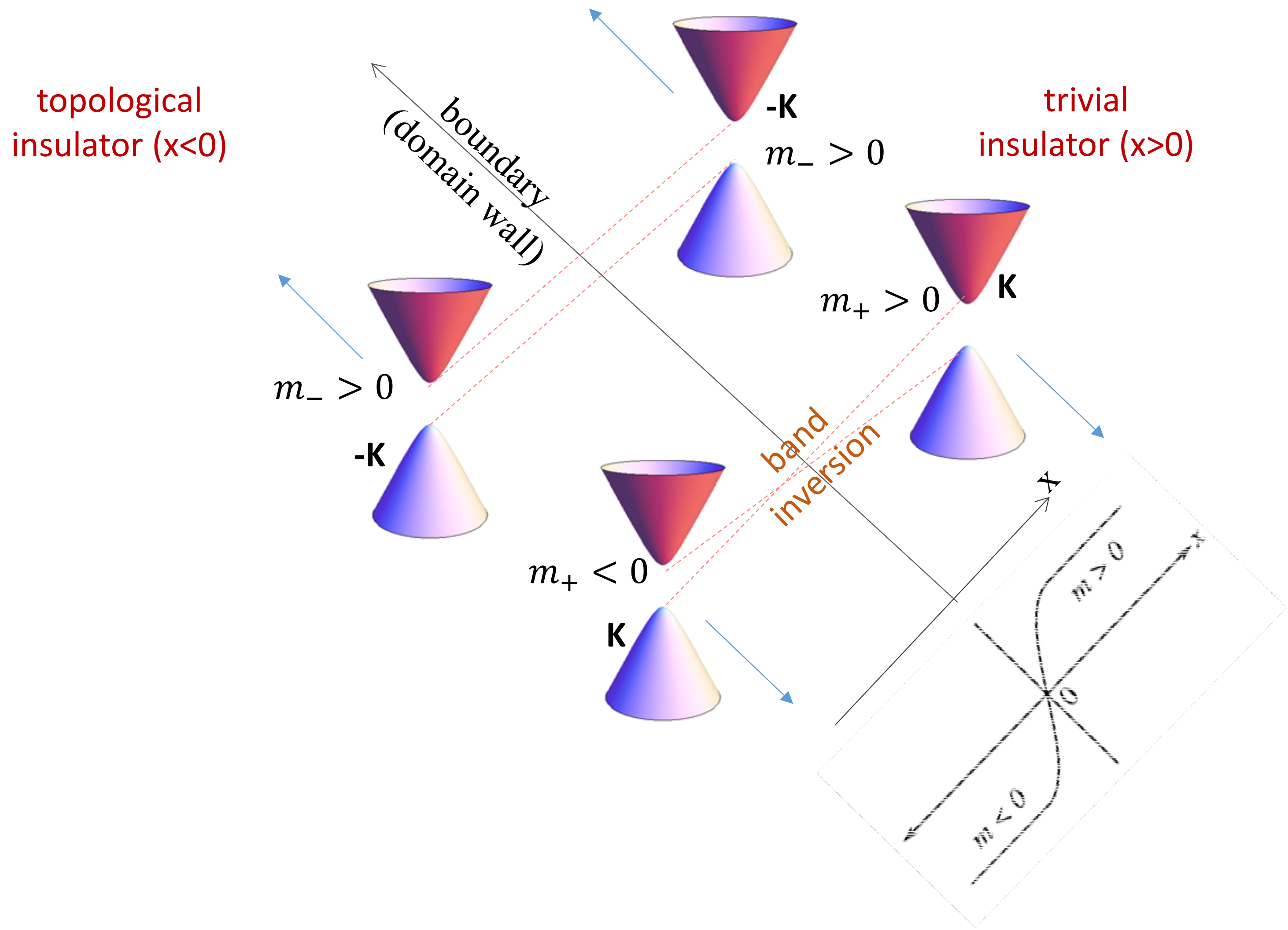
Chern number: $$n_{\mathbb{Z}} = \frac{1}{8\pi}\int_{\mathrm{BZ}} d^2\mathbf{k}\, \epsilon^{\mu\nu}\hat{\mathbf{d}} \cdot \left[\partial_{k_\mu}\hat{\mathbf{d}} \times \partial_{k_\nu}\hat{\mathbf{d}}\right]$$

Phase diagram of the Haldane model



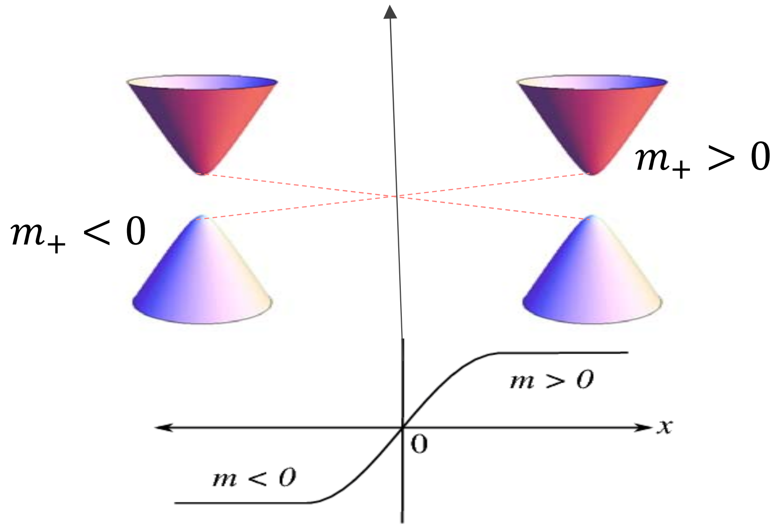


Bulk-boundary correspondence: Application to the Haldane model



Domain wall along the x-axis

Dispersing Jackiw-Rebbi-like edge modes



k_y conserved

$$\mathcal{H}_+ = v_F\left(-i\hat{\sigma}_x\partial_x + i\hat{\sigma}_y k_y\right) + m_+(x)\hat{\sigma}_z$$

Fixing k_y maps the problem on the 1D Jackiw-Rebbi model, with the edge mode

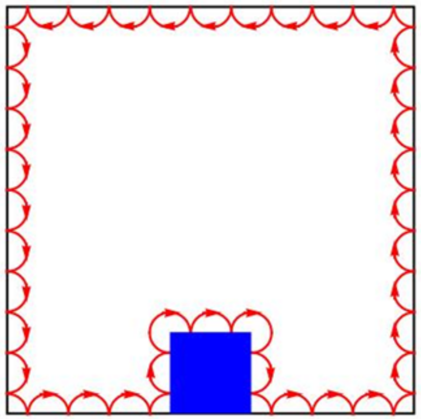
$$|\psi(k_y)\rangle = e^{ik_y y}\ exp\left[-\frac{1}{v_F}\int_{-\infty}^{x}|m_+(x')|dx'\right]|\chi_+\rangle \quad \text{where} \quad |\chi_+\rangle = \begin{pmatrix}1\\ i\end{pmatrix}$$

$$\mathcal{H}_+|\psi(k_y)\rangle = v_F k_y|\psi(k_y)\rangle$$

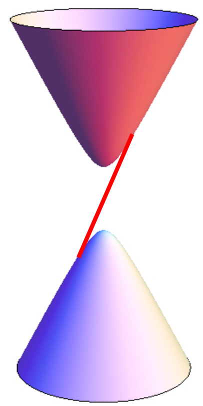
CHIRAL STATE

Properties of the chiral edge mode

$$|\psi(k_y)\rangle = e^{ik_y y}\ exp\left[-\frac{1}{v_F}\int_{-\infty}^{x}|m_+(x')|dx'\right]|\chi_+\rangle$$



Conducting chiral edge



- The chiral mode can not be stopped by any obstacle or edge disorder.
- Normally, any 1D system localizes at low temperature (Anderson insulator). The chiral edge is protected from localization.
- Such a 1D mode can not appear in a pure 1D system, only at a boundary of a higher-dimensional system.
- The chiral edge carries the quantized Hall conductivity (IQHE). $\sigma_{xy}=j_x/E_y= n\, e^2/h$

III) A brief incursion into 2D topological insulators Or The 2D spin quantum Hall insulator

Destroying Dirac points in spinfull graphene

Graphene Hamiltonian with spin & valley indices restored

$$\mathcal{H} = \mathrm{v_F}\left(\hat{\mathbb{I}}\otimes\hat{\tau}_z\otimes\hat{\sigma}_x\, q_x + \hat{\mathbb{I}}\otimes\hat{\mathbb{I}}\otimes\hat{\sigma}_y\, q_y\right) + \hat{V}$$

Spin Valleys (K & K') Sublattices (A & B) gap-opening perturbation

1. Inversion (P-) breaking perturbation (trivial insulator, e.g. Boron nitride)

$$\hat{V} = m_p\hat{\mathbb{I}}\otimes\hat{\mathbb{I}}\otimes\hat{\sigma}_z$$

2. T-reversal breaking perturbation (Chern insulator, e.g. Haldane model)

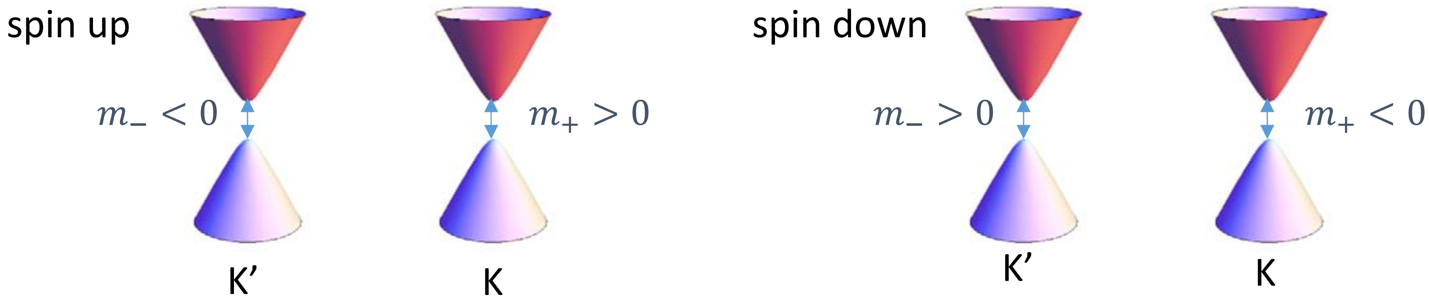
$$\hat{V} = m_T\hat{\mathbb{I}}\otimes\hat{\tau}_z\otimes\hat{\sigma}_z$$

3. Symmetry preserving perturbation (topological insulator, Kane-Mele model)

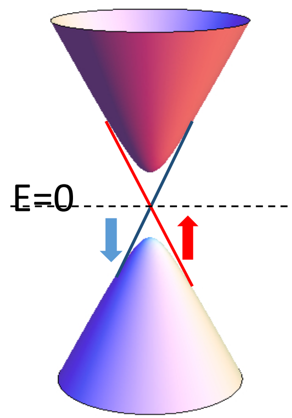
$$\hat{V} = m_{SO}\hat{S}_z\otimes\hat{\tau}_z\otimes\hat{\sigma}_z$$

The Kane-Mele model

Kane-Mele model = Haldane model ^{2}



$$\breve{\mathcal{H}}_{Kane-Mele} = \begin{pmatrix} \hat{\mathcal{H}}_{Haldane} & 0 \\ 0 & \hat{\mathcal{H}}^{*}_{Haldane} \end{pmatrix}$$



Spin-Hall conductivity:

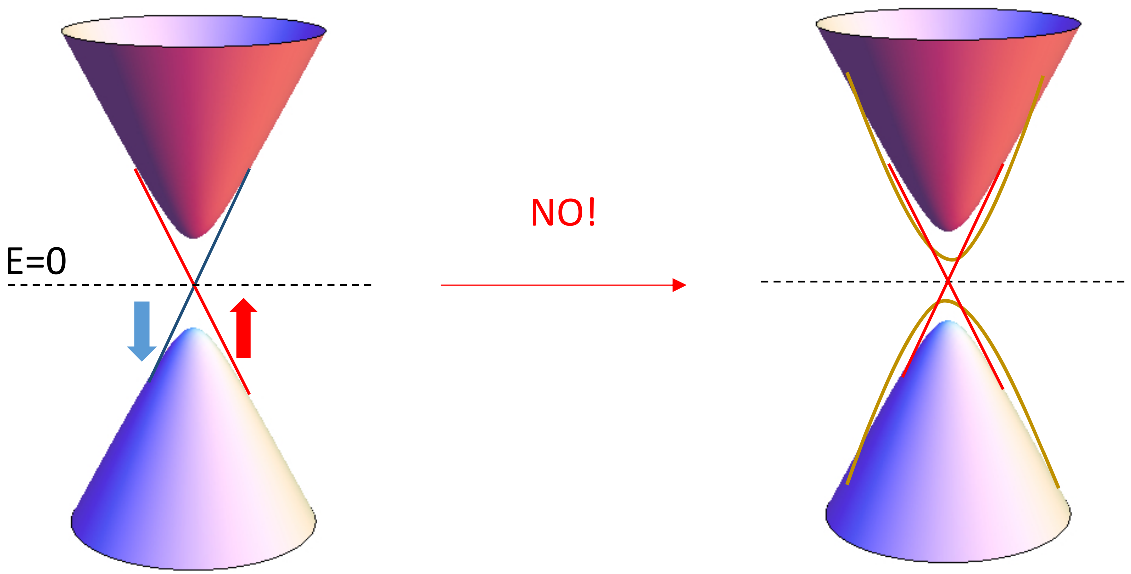
$$\sigma^s_{xy} = \sigma^{\uparrow}_{xy} - \sigma^{\downarrow}_{xy} = (n_\uparrow - n_\downarrow)\frac{e^2}{h}$$

Can the degeneracy be lifted?

- Other spin-orbit couplings are possible (e.g., Rashba), which introduce off-diagonal terms and break spin conservation (no notion of spin up or down exists)

$$\begin{pmatrix} \hat{\mathcal{H}}_{Haldane} & 0 \\ 0 & \hat{\mathcal{H}}^{*}_{Haldane} \end{pmatrix} \longrightarrow \begin{pmatrix} \hat{\mathcal{H}}_{Haldane} & \# \\ \# & \hat{\mathcal{H}}^{*}_{Haldane} \end{pmatrix}$$

- Can the generic spin-orbit perturbation lift the degeneracy?



The degeneracy is protected by T-reversal symmetry

- Time-reversal operator = spin-rotation and complex conjugation

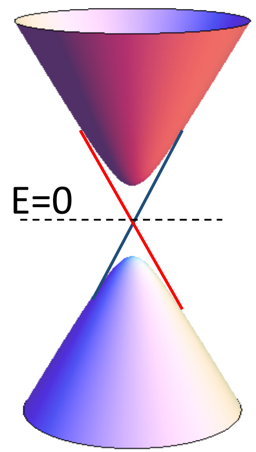
$$\mathbb{T}\begin{pmatrix}\psi_\uparrow\\ \psi_\downarrow\end{pmatrix} = e^{i\pi\hat{S}_y}\begin{pmatrix}\psi_\uparrow\\ \psi_\downarrow\end{pmatrix}^* = \begin{pmatrix}\psi_\downarrow^*\\ -\psi_\uparrow^*\end{pmatrix}, \qquad \mathbb{T}^2 = -1$$

- Time-reversal symmetry implies $[\mathcal{H}, \mathbb{T}]$=0

- This guarantees double-degeneracy of the spectrum

- Hence, the must be (at least) 2 distinct, degenerate states with energy E connected by $\mathbb{T}$-reversal (Kramers doublet).

- We can't remove degeneracy at E=0, as long as perturbation does not break $\mathbb{T}$-reversal!



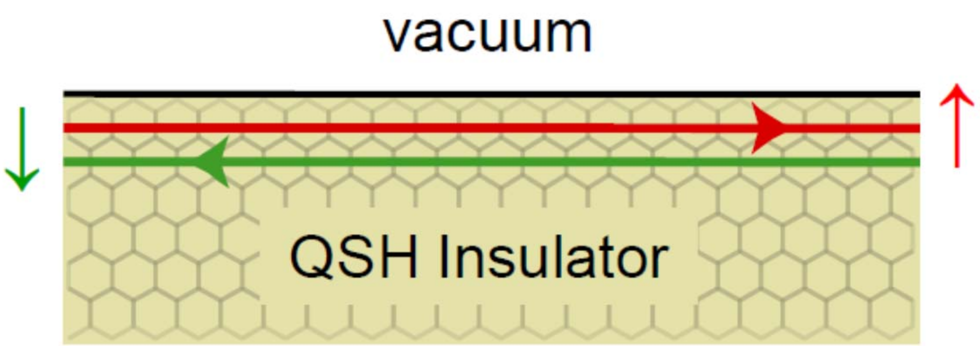
A **(non-Chern) topological invariant** is responsible for this robustness

This is the **Z2 invariant**

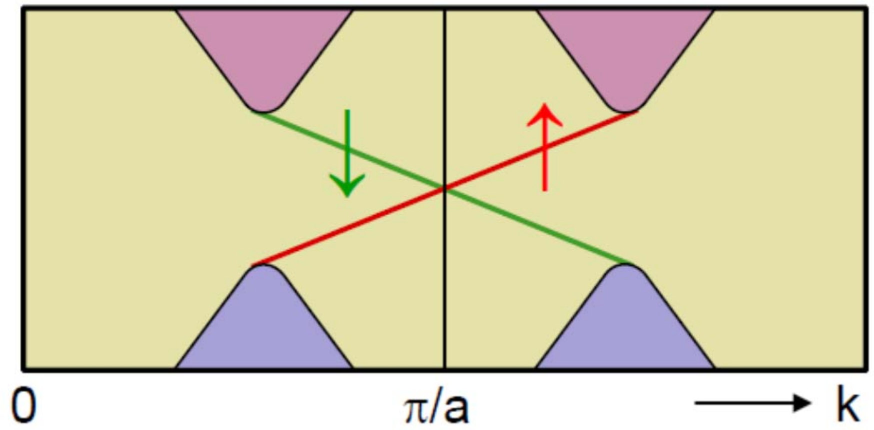
1D Helical edges states

Bulk energy gap, but gapless edge states

"Spin Filtered" or "helical" edge states



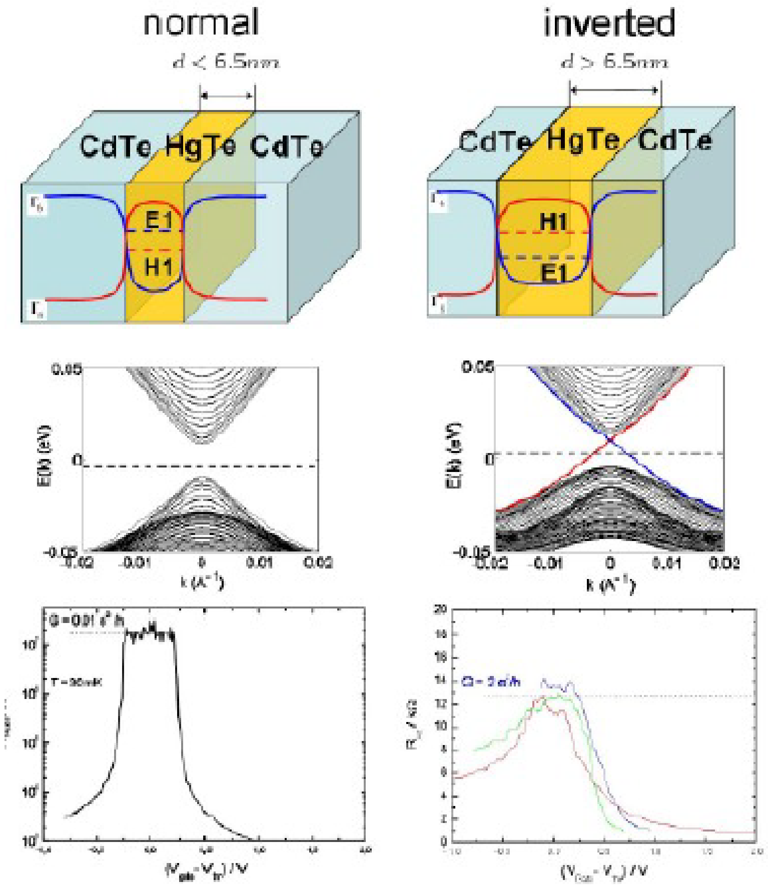
Edge band structure

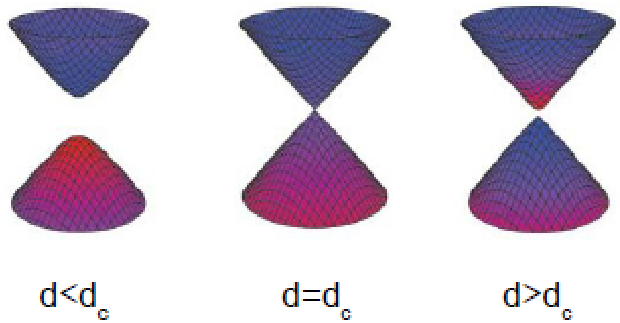


Edge states form a unique 1D electronic conductor

- HALF an ordinary 1D electron gas
- Protected by Time Reversal Symmetry

Conductance in HgTe/ CdTe heterojunctions





Konig et al. Science 2007

See also Multiterminal conductance probes (Roth et al., Science 325, 294 (2009))

Spin polarization of the quantum spin Hall edge states (Brune et al.,Nature Physics 8, 486 (2012))

See also quantum spin Hall effect in WTe_2, S. Wu et al., Science 359, 76 (2018)

IV) 2D chiral topological superconductors