Chiral edge states in topological insulators and superconductors

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Outline

0) Preliminaries of (topological) band theory

I) Integer quantum Hall effect

II) The anomalous quantum Hall effect

III) A brief incursion into 2D topological insulators

IV) 2D chiral topological superconductors

V) How about 3D?
Preliminaries of (topological) band theory
Elements of Traditional Band Theory

Non-interacting electrons moving in a perfectly periodic array of atoms

- Electron Hamiltonian commutes with lattice translations

\[ [H, T(R)] = 0 \]

\[ R = n_x a_x + n_y a_y + n_z a_z, \quad n_\alpha \text{ is an integer} \]

Crystal momentum \( k \) is conserved

- The wave vector \( k \) is defined modulo the reciprocal lattice vector (reciprocal lattice is the Fourier transform of the real-space lattice)

\[ k \sim k \mod G \]

- The wave-vector \( k \) “lives” on a \( d \)-dimensional torus

\[ k \in \mathbb{T}^d \quad (1D: -\pi/a \leq k \leq \pi/a, \text{ with the end points “glued”}) \]
Elements of Traditional Band Theory

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Lattice translation
Symmetry

\[ T(R)|\psi_k\rangle = e^{i k \cdot R} |\psi_k\rangle \]

Crystal momentum \( k \) is conserved

Bloch thm:

\[ |\psi_k\rangle = e^{i k \cdot R} |u_k\rangle \]

Bloch Hamiltonian:

\[ H(k) = e^{-i k \cdot R} H e^{i k \cdot R} \]

\[ H(k)|u_n(k)\rangle = E_n(k)|u_n(k)\rangle \]
Insulators and metals

- Bloch theorem and band structure:

\[
\left[ -\frac{\hbar^2 \nabla^2}{2m} + V(r) \right] \psi(r) = E\psi(r), \quad V(r) = V(r + a)
\]

\[
\psi_p(r) = u_p(r) e^{i \mathbf{p} \cdot \mathbf{r}}, \quad \text{with} \quad u_p(r) = u_p(r + a)
\]
Quantum topological equivalence

• How to define topological invariants for quantum states of matter?

• We need a notion of topological equivalence of quantum states.

• The notion of quantum topological equivalence follows from adiabatic continuity

If we can adiabatically deform $|0\rangle$ into $|0'\rangle$, then $|0\rangle \sim |0'\rangle$
Band topological equivalence

- How to define topological invariants for quantum states of matter?
- We need a notion of topological equivalence of quantum states.

**Topological Equivalence : adiabatic continuity**

Band structures are equivalent if they can be continuously deformed into one another without closing the energy gap.
I) The integer Quantum Hall effect
Classical Hall effect (1879)

Classical equation of motion

\[ m \left( \frac{d\vec{v}}{dt} + \frac{\vec{v}}{\tau} \right) = -e(\vec{E} + \vec{v} \times \vec{B}) \]

Conductivity tensor

\[ \sigma_{xx} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \]
\[ \sigma_{yy} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \]

Drude Conductivity

\[ \sigma_0 = \frac{ne^2 \tau}{m} \]
\[ \omega_c = \frac{eB}{m} \]

Resistivity tensor

\[ \rho_{xx} = \sigma_0^{-1} \]
\[ \rho_{yy} = \frac{m \omega_c}{ne^2} = \frac{1}{ne} B \]

\[ \rho_H = \rho_{xy} \propto B \]
Quantum Hall effect (1980)

Quantum Hall effect

- **Quantization of the Hall resistance at low temperature**:
  \[ R_H = \frac{h}{e^2} \frac{1}{n} \]

  Results independent of geometrical and microscopic details

  Quantum of resistance; UNIVERSAL constant

  Used as a metrological unit: help to redefine the unit of mass!

Quantum Hall effect

- Quantum Hall conductivity changes by plateaus.
- Each plateau is perfectly quantized by an integer number in unit of $e^2/h$.

\[ J_y = \sigma_{xy} E_x \]
\[ \sigma_{xy} = n \frac{e^2}{h} \]

Integer accurate to $10^{-9}$
Semi-classical picture

Electron in an orbital magnetic field:

\[ H = \frac{1}{2m_e} (\vec{p} + e\vec{A})^2 \]

\[ \varepsilon_n = \left(n + \frac{1}{2}\right) \hbar \omega_c \]

2D Cyclotron Motion, Landau Levels

Landau levels
Why such perfect robustness & quantization?
Semi-classical picture

- Landau levels (LLs) bend near sample edge.
- The Fermi level intersects LLs at the edge.
- Nb of edge states at the Fermi level = Nb of occupied bulk LLs

Landau levels with a bulk gap and (protected) edge states
The edges’ viewpoint: Robustness of n

- Electrons on same edge move along the same direction.
- Electrons on opposite edges move along the opposite directions.

Chirality = Consequence of time reversal symmetry breaking

Robustness against backscattering

- chiral edge state cannot be localized by disorder (no backscattering)
- edge states are therefore perfect charge conductors
The bulk point of view

The quantum Hall effect: a topological property?

Distinction between the integer quantum Hall state and a conventional insulator is a topological property of the band structure

\[ \mathcal{H}(k) : \text{Brillouin zone} \rightarrow \text{Hamiltonians with energy gap} \]

Classified by Chern number:

\[ n = \frac{i}{2\pi} \sum_{\text{filled states}} \int \mathcal{F} d^2k \quad (= \text{topological invariant}) \quad n \in \mathbb{Z} \]

Kubo formula:

\[ \sigma_{xy} = \frac{e^2}{h} \frac{i}{2\pi} \sum_{\text{filled states}} \int \mathcal{F} d^2k = \frac{e^2}{h} n \]

Thouless et al., PRL 49, 405 (1982)

Alternative description: \( n \) is a bulk topological invariant
Example of a topological invariant

Can we tell by local measurements whether we are living on the surface of a sphere or a torus?

\[ g = 0 \]

\[ g = 1 \]

\[ \int_S K_g \, dA = 4\pi (s - g) \]

Gaussian curvature, \( K_g = \pm \frac{1}{R_1 R_2} \)

**Topological invariant** = quantity that does not change under continuous deformation
Berry connection & curvature

For a given band, we can introduce:

**Berry connection:**

\[ A(k) = -\text{Im} \langle u_k | \nabla_k | u_k \rangle \]

**Berry phase:**

\[ \phi = \oint A(k) \cdot dk \]

**Berry curvature**

\[ \Omega(k) = \nabla \times A \]

\[ \Omega_z(k) = -2\text{Im} \left( \left\langle \frac{du}{dk_x} \right| \frac{du}{dk_y} \right) \]

**Stokes thm:**

\[ \phi = \int \Omega_z(k) d^2k \]
Chern theorem

Berry curvature

\( (F \equiv \Omega) \)

Stokes thm applied to A:

\[ \phi = \int_A \mathcal{F}(\lambda) \, dS_\lambda \mod 2\pi \]

Stokes applied to B:

\[ \phi = -\int_B \mathcal{F}(\lambda) \, dS_\lambda \mod 2\pi \]

Subtract:

\[ 0 = \oint \mathcal{F}(\lambda) \, dS_\lambda \mod 2\pi \]

Chern Theorem:

\[ \oint \mathcal{F}(\lambda) \, dS_\lambda = 2\pi C \]

with \( C \in \mathbb{Z} \)

\( C = \) First Chern number
Application of Chern theorem

Let us apply this result to the Brillouin zone
Application of Chern theorem

Let us apply this result to the Brillouin zone

\[ \Omega_z(k) = -2\text{Im} \left\langle \frac{du}{dk_x} \left| \frac{du}{dk_y} \right. \right\rangle \]

\[ \phi = \int_{\text{BZ}} \Omega_z(k) \, d^2k = 2\pi C \]

Anomalous Hall conductivity:

\[ \sigma_{xy} = \frac{e^2}{h} C \]
Topological phase transition
Bulk-edge correspondence

Topological Phase 1

Topological Phase 2

Something special at the boundary
Bulk-edge correspondence

Country A
Drive on left
Hong Kong

Country B
Drive on right
China

Something special at boundary?
Bulk-edge correspondence
Bulk-edge correspondence

Two materials described by different topological invariants $C_1$ and $C_2$ placed in contact $\Rightarrow$ emergence of $|C_1 - C_2|$ gapless edge modes

$|C_1 - C_2|$ gapless edge modes
Chiral edges states in the QHE

Gapless states must exist at the interface between different topological phases

Edge states \(\sim\) skipping orbits

Smooth transition: band inversion

**Edge States**

- Empty Band
- Filled Band

\(n = 0\)
II) The anomalous quantum Hall effect or the 2D Chern insulator
Anomalous Hall effect (1881)

Measure of Hall conductivity in absence of a magnetic field
Quantum anomalous Hall effect (2013 ?)

Anomalous Hall conductivity

Like integer quantum Hall effect, but no $B_{\text{ext}}$
Quantum anomalous Hall effect (2013?)

Edge states: 2D QAH insulator

Existence of a chiral edge state without magnetic field!
Edge states: 2D QAH insulator

A. J. Bestwick et al., PRL 114, 187201 (2015)
Proof of principle: the Haldane model

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance $\sigma^{xy}$ in the absence of an external magnetic field. Massless fermions without spectral doubling occur at critical values of the model parameters, and exhibit the so-called “parity anomaly” of (2+1)-dimensional field theories.
Graphene

One orbital per site

Two atoms per unit cell (A and B)

Spinless

Band structure near Dirac cones

Emergence of massless Dirac fermions at low energies:

\[ h(k) = d(k) \cdot \sigma \]

\[ h(q) \approx v\tau z \sigma^x q_x + v\sigma^y q_y \]

Momentum measured from Dirac node
Symmetries of graphene

- Inversion symmetry A sublattice $\leftrightarrow$ B sublattice

\[ \hat{P} = \sigma_x \tau_x \]
\[ \hat{P} h(q) \hat{P} = h(-q) \]

- Time reversal symmetry:

\[ \hat{T} = \tau_x \mathcal{K} \]
\[ \hat{T} h(q) \hat{T} = h(-q) \]
Making graphene insulating

\[ h(q) = v \tau^z \sigma^x q_x + v \sigma^y q_y + d_z(q) \sigma^z \]

**Need to break either time-reversal symmetry or inversion symmetry**

(i) **Break inversion symmetry**

\[ d_z(q) = m_s \]

Semenoff insulator (1984)

(ii) **Break time-reversal symmetry**

\[ d_z(q) = m_H \tau^z \]

Haldane insulator (1988)

= Quantum spin Hall insulator

= Chern insulator
Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the “Parity Anomaly”

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(Received 16 September 1987)

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance $\sigma^{xy}$ in the absence of an external magnetic field. Massless fermions without spectral doubling occur at critical values of the model parameters, and exhibit the so-called “parity anomaly” of (2+1)-dimensional field theories.
Topological characterization

\begin{equation*}
    h(k) = \mathbf{d}(k) \cdot \sigma
\end{equation*}

Two strategies:

i) Compute the eigenvectors, Berry connection, Berry phase and Chern number.

ii) Look at \( \mathbf{d}(k) \)

\[ E_{\pm} = \pm |\mathbf{d}(k)| \]

Spectrum flattening

\[ \hat{\mathbf{d}}(k) = \frac{\mathbf{d}(k)}{|\mathbf{d}(k)|} \]

Mapping:

\( \hat{\mathbf{d}}(k) : \text{Brillouin zone} \quad \rightarrow \quad \hat{\mathbf{d}}(k) \in S^2 \)

\[ \pi_2(S^2) = \mathbb{Z} \]
Topological characterization

\[ h(k) = d(k) \cdot \sigma \]

Two strategies:

i) Compute the eigenvectors, Berry connection, Berry phase and Chern number.

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Spectrum flattening

\[ \hat{d}(k) = \frac{d(k)}{|d(k)|} \]

Semenoff insulator

Haldane insulator

Trivial insulator: \( m_K = m_{K'} \)

Haldane insulator: \( m_K = -m_{K'} \)
Topological characterization

\[ h(k) = d(k) \cdot \sigma \]

Two strategies:

i) Compute the eigenvectors, Berry connection, Berry phase and Chern number.

ii) Look at \( d(k) \)

\[
E_{\pm} = \pm |d(k)|
\]

Spectrum flattening

\[
\hat{d}(k) = \frac{d(k)}{|d(k)|}
\]

Chern number:

\[
n_Z = \frac{1}{8\pi} \int_{BZ} \epsilon^{\mu\nu} \hat{d} \cdot \left[ \partial_{k\mu} \hat{d} \times \partial_{k\nu} \hat{d} \right]
\]

trivial phase: \( n_Z = 0 \)

non-trivial phase: \( n_Z = \pm 1 = \text{sgn}(m_H) \)
Phase diagram of the Haldane model
Bulk-boundary correspondence: Application to the Haldane model

topological insulator \((x<0)\)

trivial insulator \((x>0)\)

Domain wall along the \(x\)-axis
Dispersing Jackiw-Rebbi-like edge modes

\[ \mathcal{H}_+ = v_F (-i \hat{\sigma}_x \partial_x + i \hat{\sigma}_y k_y) + m_+(x) \hat{\sigma}_z \]

Fixing $k_y$ maps the problem on the 1D Jackiw-Rebbi model, with the edge mode

\[ |\psi(k_y)\rangle = e^{ik_y y} \exp \left[-\frac{1}{v_F} \int_{-\infty}^{x} |m_+(x')| dx' \right] |\chi_+\rangle \]

where

\[ |\chi_+\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix} \]

\[ \mathcal{H}_+ |\psi(k_y)\rangle = v_F k_y |\psi(k_y)\rangle \quad \rightarrow \quad \text{CHIRAL STATE} \]
Properties of the chiral edge mode

\[ |\psi(k_y)\rangle = e^{ik_y y} \exp \left[-\frac{1}{v_F} \int_{-\infty}^{x} |m_+(x')| dx' \right] |\chi_+\rangle \]

Conducting chiral edge

- The chiral mode can not be stopped by any obstacle or edge disorder.
- Normally, any 1D system localizes at low temperature (Anderson insulator). The chiral edge is protected from localization.
- Such a 1D mode can not appear in a pure 1D system, only at a boundary of a higher-dimensional system.
- The chiral edge carries the quantized Hall conductivity (IQHE). \[ \sigma_{xy} = j_x / E_y = n \frac{e^2}{h} \]
III) *A brief incursion into 2D topological insulators*  
*Or*  
*The 2D spin quantum Hall insulator*
Destroying Dirac points in spinfull graphene

Graphene Hamiltonian with spin & valley indices restored

\[ \mathcal{H} = v_F \left( \hat{I} \otimes \hat{z} \otimes \hat{\sigma}_x \ q_x + \hat{I} \otimes \hat{I} \otimes \hat{\sigma}_y \ q_y \right) + \hat{\mathcal{V}} \]

1. Inversion (P-) breaking perturbation (trivial insulator, e.g. Boron nitride)

\[ \hat{\mathcal{V}} = m_p \hat{I} \otimes \hat{I} \otimes \hat{\sigma}_z \]

2. T-reversal breaking perturbation (Chern insulator, e.g. Haldane model)

\[ \hat{\mathcal{V}} = m_T \hat{I} \otimes \hat{z} \otimes \hat{\sigma}_z \]

3. Symmetry preserving perturbation (topological insulator, Kane-Mele model)

\[ \hat{\mathcal{V}} = m_{SO} \hat{z} \otimes \hat{z} \otimes \hat{\sigma}_z \]
The Kane-Mele model

Kane-Mele model = Haldane model

Spin-Hall conductivity:

\[ \sigma_{xy}^s = \sigma_{xy}^\uparrow - \sigma_{xy}^\downarrow = (n_\uparrow - n_\downarrow) \frac{e^2}{h} \]
Can the degeneracy be lifted?

- Other spin-orbit couplings are possible (e.g., Rashba), which introduce off-diagonal terms and break spin conservation (no notion of spin up or down exists)

\[
\begin{pmatrix}
\hat{H}_{\text{Haldane}} & 0 \\
0 & \hat{H}_{\text{Haldane}}^*
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\hat{H}_{\text{Haldane}} & \# \\
\# & \hat{H}_{\text{Haldane}}^*
\end{pmatrix}
\]

- Can the generic spin-orbit perturbation lift the degeneracy?

\[E=0\]
The degeneracy is protected by T-reversal symmetry

- Time-reversal operator = spin-rotation and complex conjugation

\[ T \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = e^{i\pi s_y} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}^* = \begin{pmatrix} \psi_\downarrow^* \\ -\psi_\uparrow^* \end{pmatrix}, \quad T^2 = -1 \]

- Time-reversal symmetry implies \([\mathcal{H}, T] = 0\)

- This guarantees double-degeneracy of the spectrum

- Hence, the must be (at least) 2 distinct, degenerate states with energy \(E\) connected by \(T\)-reversal (Kramers doublet).

- We can’t remove degeneracy at \(E=0\), as long as perturbation does not break \(T\)-reversal!

A (non-Chern) topological invariant is responsible for this robustness

This is the \(\mathbb{Z}_2\) invariant
1D Helical edges states

Bulk energy gap, but gapless edge states

“Spin Filtered” or “helical” edge states

Edge band structure

Edge states form a unique 1D electronic conductor

• HALF an ordinary 1D electron gas
• Protected by Time Reversal Symmetry
Conductance in HgTe/ CdTe heterojunctions

See also Multiterminal conductance probes (Roth et al., Science 325, 294 (2009))
Spin polarization of the quantum spin Hall edge states (Brune et al., Nature Physics 8, 486 (2012))
See also quantum spin Hall effect in WTe₂, S. Wu et al., Science 359, 76 (2018)
IV) 2D chiral topological superconductors